An Introduction to the Geometry of Null Hypersurfaces in General Relativity

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Abstract

These notes are based on a mini-course delivered at the 2026 Summer School on Geometry and Analysis at the Institute of Geometry and Physics, University of Science and Technology of China (Hefei). Their purpose is to provide an introduction to the differential geometry of null hypersurfaces in general relativity, with an emphasis on the underlying geometric structures rather than the analytic estimates. We follow chapter one and two of [1], and [2].

The topics include the double-null formalism, Ricci coefficients and null curvature components, the null structure equations, the characteristic initial value problem, the description of the free characteristic data through the conformal geometry of the sections, and the geometric mechanism underlying Luk's local existence theorem. Throughout, the goal is to present the geometric framework that forms the foundation of modern mathematical relativity.

Contents

1 The geometric setting

We use Christodoulou's double-null notation. The spacetime connection is ∇, the intrinsic connection of $S_{u,\underline{u}}$ is $\nabla\!\!\!\!/$, and $D, \underline{D}$ denote the projected Lie derivatives along $L, \underline{L}$.

1.1 Optical functions and null vectorfields

Let $(\mathcal{M}, \mathbf{g})$ be a four-dimensional Lorentzian manifold. We use three normalizations of the same null directions: $L', \underline{L}'$ are affine geodesic generators, e_4, e_3 are normalized by $\mathbf{g}(e_3, e_4) = -2$, and $L, \underline{L}$ are adapted to the optical parameters $u, \underline{u}$.

Definition 1.1 (Optical functions and null hypersurfaces)**.** The functions $u, \underline{u}$ are optical functions if

$$\mathbf{g}^{-1}(du, du) = 0, \qquad \mathbf{g}^{-1}(d\underline{u}, d\underline{u}) = 0.$$

Set

$$C_u = \{u = \text{const}\}, \qquad \underline{C}_{\underline{u}} = \{\underline{u} = \text{const}\}, \qquad S_{u,\underline{u}} = C_u \cap \underline{C}_{\underline{u}}.$$

The condition $\mathbf{g}^{-1}(du, du) = 0$ means that the normal covector du has null metric dual. Thus

$$\mathbf{g}^{-1}(du) \in TC_u, \qquad \mathbf{g}^{-1}(d\underline{u}) \in T\underline{C}_{\underline{u}},$$

because

$$du(\mathbf{g}^{-1}(du)) = \mathbf{g}^{-1}(du, du) = 0, \qquad d\underline{u}(\mathbf{g}^{-1}(d\underline{u})) = \mathbf{g}^{-1}(d\underline{u}, d\underline{u}) = 0.$$

This is the basic distinction between spacelike and null hypersurfaces: for a null hypersurface, the normal direction is also tangent. Hence C_u and $\underline{C}_{\underline{u}}$ are foliated by null curves, called their null generators.

Definition 1.2 (Null generators)**.** Define

$$L' = -2\mathbf{g}^{-1}(du), \qquad \underline{L}' = -2\mathbf{g}^{-1}(d\underline{u}), \qquad -\mathbf{g}(L', \underline{L}') = 2\Omega^{-2}.$$

Then define

$$e_4 = \Omega L', \qquad e_3 = \Omega \underline{L}', \qquad L = \Omega e_4, \qquad \underline{L} = \Omega e_3.$$

Thus

$$\mathbf{g}(e_3, e_4) = -2, \qquad L = \Omega^2 L', \qquad \underline{L} = \Omega^2 \underline{L}'.$$

Here L', e_4, and L are three rescalings of the same null direction tangent to C_u. Similarly, $\underline{L}'$, e_3, and $\underline{L}$ are three rescalings of the same null direction tangent to $\underline{C}_{\underline{u}}$.

Lemma 1.3 (Basic properties of the null generators)**.** *The fields L' and $\underline{L}'$ are null, tangent to $C_u, \underline{C}_{\underline{u}}$, and affinely geodesic:*

$$\nabla_{L'} L' = 0, \qquad \nabla_{\underline{L}'} \underline{L}' = 0.$$

Moreover,

$$Lu = 0, \quad L\underline{u} = 1, \qquad \underline{L}u = 1, \quad \underline{L}\underline{u} = 0, \qquad \mathbf{g}(L, \underline{L}) = -2\Omega^2.$$

Consequently L is tangent to C_u, $\underline{L}$ is tangent to $\underline{C}_{\underline{u}}$, and

$$L : S_{u,\underline{u}} \longrightarrow S_{u,\underline{u}+t}, \qquad \underline{L} : S_{u,\underline{u}} \longrightarrow S_{u+t,\underline{u}}.$$

Proof. The eikonal equations give

$$\mathbf{g}(L', L') = 4\mathbf{g}^{-1}(du, du) = 0, \qquad \mathbf{g}(\underline{L}', \underline{L}') = 4\mathbf{g}^{-1}(d\underline{u}, d\underline{u}) = 0,$$

and

$$L'u = -2\mathbf{g}^{-1}(du, du) = 0, \qquad \underline{L}'\underline{u} = -2\mathbf{g}^{-1}(d\underline{u}, d\underline{u}) = 0.$$

Thus $L' \in TC_u$ and $\underline{L}' \in T\underline{C}_{\underline{u}}$. Also,

$$\begin{aligned} \mathbf{g}_{\lambda\mu} L'^{\nu} \nabla_\nu L'^{\mu} &= -2L'^{\nu} \nabla_\nu \partial_\lambda u = -2L'^{\nu} \nabla_\lambda \partial_\nu u \\ &= 4\mathbf{g}^{\nu\kappa} \partial_\kappa u \, \nabla_\lambda \partial_\nu u = 2\partial_\lambda \big(\mathbf{g}^{\nu\kappa} \partial_\nu u \, \partial_\kappa u\big) = 0. \end{aligned}$$

Hence $\nabla_{L'} L' = 0$. The same computation with $\underline{u}$ gives $\nabla_{\underline{L}'} \underline{L}' = 0$. Finally,

$$\mathbf{g}(L', \underline{L}') = 4\mathbf{g}^{-1}(du, d\underline{u}) = -2\Omega^{-2},$$

so

$$L'\underline{u} = \Omega^{-2}, \qquad \underline{L}'u = \Omega^{-2}.$$

Multiplying by Ω^2,

$$Lu = 0, \quad L\underline{u} = 1, \qquad \underline{L}u = 1, \quad \underline{L}\underline{u} = 0,$$

and

$$\mathbf{g}(L, \underline{L}) = \Omega^4 \mathbf{g}(L', \underline{L}') = -2\Omega^2.$$

The final statement follows from $Lu = 0$, $L\underline{u} = 1$, $\underline{L}u = 1$, and $\underline{L}\underline{u} = 0$. □

Remark 1.4 (Geometric meaning)*.* The three normalizations serve different purposes:

- $L', \underline{L}'$ are the affine geodesic generators:

$$\nabla_{L'} L' = 0, \qquad \nabla_{\underline{L}'} \underline{L}' = 0.$$

- e_4, e_3 are the normalized null frame:

$$\mathbf{g}(e_3, e_4) = -2.$$

  This is the frame used to define Ricci coefficients and curvature components.

- $L, \underline{L}$ are adapted to the optical parameters:

$$L\underline{u} = 1, \qquad \underline{L}u = 1.$$

  Thus L carries $S_{u,\underline{u}}$ to $S_{u,\underline{u}+t}$, while $\underline{L}$ carries $S_{u,\underline{u}}$ to $S_{u+t,\underline{u}}$.

Remark 1.5 (Riemannian comparison). For a Riemannian hypersurface the normal direction is transverse. For a null hypersurface the normal direction is tangent. Thus C_u and $\underline{C}_{\underline{u}}$ have degenerate induced metrics, and the Riemannian geometry is carried by $S_{u,\underline{u}}$.

1.2 $S_{u,\underline{u}}$-geometry

Definition 1.6 (S-tensorfields and intrinsic operations)**.** An S-tensorfield is tangent to $S_{u,\underline{u}}$ in all its entries. The induced metric, connection, Gauss curvature, and area form are

$$\not{g} = \mathbf{g}|_{S_{u,\underline{u}}}, \qquad \not{\nabla} = \nabla^{\not{g}}, \qquad K = K(\not{g}), \qquad \not{\epsilon} = \text{area form of } \not{g}.$$

For a function f, an S-one-form ξ, and a symmetric S-two-tensor θ, set

$$(\not{d}f)_A = e_A(f), \qquad \mathrm{d\!\!/iv}\xi = \not{g}^{AB}\not{\nabla}_A\xi_B, \qquad \mathrm{cu\!\!/rl}\xi = \not{\epsilon}^{AB}\not{\nabla}_A\xi_B,$$

$$\mathrm{tr}\theta = \not{g}^{AB}\theta_{AB}, \qquad |\theta|^2 = \not{g}^{AC}\not{g}^{BD}\theta_{AB}\theta_{CD},$$

and

$$(\not{\nabla}\widehat{\otimes}\xi)_{AB} = \not{\nabla}_A\xi_B + \not{\nabla}_B\xi_A - (\mathrm{d\!\!/iv}\xi)\not{g}_{AB}.$$

Lemma 1.7 (Projection and trace-free symmetrization)**.** *For S-vectorfields X, Y,*

$$\not{\nabla}_X Y = (\nabla_X Y)^\top.$$

Moreover,

$$\mathrm{tr}(\not{\nabla}\widehat{\otimes}\xi) = 0.$$

Proof. For any S-vectorfield Z,

$$\begin{aligned} X\not{g}(Y,Z) &= X\mathbf{g}(Y,Z) = \mathbf{g}(\nabla_X Y, Z) + \mathbf{g}(Y, \nabla_X Z) \\ &= \not{g}((\nabla_X Y)^\top, Z) + \not{g}(Y, (\nabla_X Z)^\top). \end{aligned}$$

Thus $(\nabla_X Y)^\top$ is the Levi-Civita connection of $\not{g}$. Also,

$$\not{g}^{AB}(\not{\nabla}\widehat{\otimes}\xi)_{AB} = 2\mathrm{d\!\!/iv}\xi - (\mathrm{d\!\!/iv}\xi)\not{g}^{AB}\not{g}_{AB} = 0.$$

□

Definition 1.8 (Projected Lie derivatives)**.** For an S-tensorfield θ,

$$D\theta = (\mathcal{L}_L\theta)^\top, \qquad \underline{D}\theta = (\mathcal{L}_{\underline{L}}\theta)^\top.$$

For functions,

$$Df = Lf, \qquad \underline{D}f = \underline{L}f.$$

Lemma 1.9 (Commutation with $\not{d}$)**.** *For any function f,*

$$D\not{d}f = \not{d}Df, \qquad \underline{D}\not{d}f = \not{d}\,\underline{D}f.$$

Proof. If X is tangent to $S_{u,\underline{u}}$, then

$$[L,X]u = L(Xu) - X(Lu) = 0, \qquad [L,X]\underline{u} = L(X\underline{u}) - X(L\underline{u}) = 0.$$

Thus $[L,X]$ is again tangent to $S_{u,\underline{u}}$. Hence

$$\begin{aligned}(D\not{d}f)(X) &= L((\not{d}f)(X)) - (\not{d}f)([L,X]) = L(Xf) - [L,X]f \\ &= X(Lf) = (\not{d}Df)(X).\end{aligned}$$

The $\underline{D}$-identity is identical. □

Remark 1.10 (Transported S-vectors). When deriving propagation equations, one extends $X, Y \in T_p S_{u,\underline{u}}$ along the L-generator by

$$[L,X] = 0, \qquad [L,Y] = 0.$$

Then X, Y remain tangent to the moving sections $S_{u,\underline{u}}$, and differentiating $\theta(X,Y)$ along L produces no commutator terms. The same convention is used along $\underline{L}$.

1.3 Null second fundamental forms

Definition 1.11 (Null second fundamental forms)**.** Define

$$\chi_{AB} = \mathbf{g}(\nabla_A e_4, e_B), \qquad \underline{\chi}_{AB} = \mathbf{g}(\nabla_A e_3, e_B).$$

Lemma 1.12 (Basic properties of $\chi, \underline{\chi}$)**.** *The tensors $\chi, \underline{\chi}$ are symmetric, and*

$$\mathcal{L}_{e_4}\not{g} = 2\chi, \qquad \mathcal{L}_{e_3}\not{g} = 2\underline{\chi}.$$

Consequently,

$$D\not{g} = 2\Omega\chi, \qquad \underline{D}\not{g} = 2\Omega\underline{\chi}.$$

Proof. Since e_4 is normal to C_u,

$$\begin{aligned}\chi_{AB} - \chi_{BA} &= \mathbf{g}(\nabla_A e_4, e_B) - \mathbf{g}(\nabla_B e_4, e_A) \\ &= -\mathbf{g}(e_4, \nabla_A e_B - \nabla_B e_A) = -\mathbf{g}(e_4, [e_A, e_B]) = 0.\end{aligned}$$

The proof for $\underline{\chi}$ is the same. Also,

$$\begin{aligned}(\mathcal{L}_{e_4}\not{g})_{AB} &= e_4\mathbf{g}(e_A, e_B) - \mathbf{g}([e_4, e_A], e_B) - \mathbf{g}(e_A, [e_4, e_B]) \\ &= \mathbf{g}(\nabla_A e_4, e_B) + \mathbf{g}(e_A, \nabla_B e_4) = 2\chi_{AB}.\end{aligned}$$

The e_3-formula is identical. Since $L = \Omega e_4$, $\underline{L} = \Omega e_3$, and e_3, e_4 are normal to $S_{u,\underline{u}}$, the formulas for $D\not{g}, \underline{D}\not{g}$ follow. □

Lemma 1.13 (Area variation)**.**

$$e_4(d\mu_{\not{g}}) = (\mathrm{tr}\chi)d\mu_{\not{g}}, \qquad e_3(d\mu_{\not{g}}) = (\mathrm{tr}\underline{\chi})d\mu_{\not{g}}.$$

Thus

$$D(d\mu_{\not{g}}) = \Omega(\mathrm{tr}\chi)d\mu_{\not{g}}, \qquad \underline{D}(d\mu_{\not{g}}) = \Omega(\mathrm{tr}\underline{\chi})d\mu_{\not{g}}.$$

Proof.

$$e_4(d\mu_{\not{g}}) = \frac{1}{2}\not{g}^{AB}(\mathcal{L}_{e_4}\not{g})_{AB}d\mu_{\not{g}} = (\mathrm{tr}\chi)d\mu_{\not{g}}.$$

The e_3-identity is identical, and the $D, \underline{D}$-identities follow from $L = \Omega e_4$, $\underline{L} = \Omega e_3$. □

Definition 1.14 (Expansion and shear)**.** Write

$$\chi = \hat{\chi} + \frac{1}{2}(\mathrm{tr}\chi)\not{g}, \qquad \underline{\chi} = \hat{\underline{\chi}} + \frac{1}{2}(\mathrm{tr}\underline{\chi})\not{g}.$$

The traces $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}$ are the null expansions, and $\hat{\chi}, \hat{\underline{\chi}}$ are the null shears.

Remark 1.15 (Area and conformal geometry). The expansions measure the change of area. The shears measure the change of conformal geometry. Indeed, write $\not{g} = r^2\widehat{\not{g}}$, with $\widehat{\not{g}}$ normalized by a fixed determinant condition. Then

$$\mathcal{L}_{e_4}\not{g} = 2e_4(\log r)\not{g} + r^2\mathcal{L}_{e_4}\widehat{\not{g}},$$
$$\mathcal{L}_{e_3}\not{g} = 2e_3(\log r)\not{g} + r^2\mathcal{L}_{e_3}\widehat{\not{g}}.$$

Since $\mathcal{L}_{e_4}\widehat{\not{g}}$ and $\mathcal{L}_{e_3}\widehat{\not{g}}$ are trace-free,

$$\mathrm{tr}\chi = 2e_4(\log r), \qquad \hat{\chi} = \frac{1}{2}r^2\mathcal{L}_{e_4}\widehat{\not{g}},$$

and

$$\mathrm{tr}\underline{\chi} = 2e_3(\log r), \qquad \hat{\underline{\chi}} = \frac{1}{2}r^2\mathcal{L}_{e_3}\widehat{\not{g}}.$$

Thus $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}$ control the scale of $\not{g}$, while $\hat{\chi}, \hat{\underline{\chi}}$ control the conformal class of $\not{g}$.

Remark 1.16 (Riemannian comparison). A Riemannian hypersurface has one second fundamental form. The spacelike surface $S_{u,\underline{u}}$ has two null second fundamental forms, one in each null normal direction.

1.4 Torsion forms and lapse coefficients

We write

$$\nabla_A = \nabla_{e_A}, \qquad \nabla_4 = \nabla_{e_4}, \qquad \nabla_3 = \nabla_{e_3}.$$

Definition 1.17 (Torsion forms and lapse coefficients)**.** Define

$$\zeta_A = \frac{1}{2}\mathbf{g}(\nabla_A e_4, e_3),$$

$$\eta_A = \frac{1}{2}\mathbf{g}(\nabla_3 e_4, e_A) = -\frac{1}{2}\mathbf{g}(\nabla_3 e_A, e_4),$$
$$\underline{\eta}_A = \frac{1}{2}\mathbf{g}(\nabla_4 e_3, e_A) = -\frac{1}{2}\mathbf{g}(\nabla_4 e_A, e_3),$$

and

$$\omega = -\frac{1}{4}\mathbf{g}(\nabla_4 e_3, e_4), \qquad \underline{\omega} = -\frac{1}{4}\mathbf{g}(\nabla_3 e_4, e_3).$$

Remark 1.18 (Dictionary with Christodoulou's lapse notation). Christodoulou defines the lapse derivatives by

$$\omega_{\mathrm{Chr}} = D\log\Omega, \qquad \underline{\omega}_{\mathrm{Chr}} = \underline{D}\log\Omega.$$

With the present connection-coefficient convention,

$$D\log\Omega = -2\Omega\omega, \qquad \underline{D}\log\Omega = -2\Omega\underline{\omega}.$$

Equivalently,

$$\omega = -\frac{1}{2}\Omega^{-1}\omega_{\mathrm{Chr}}, \qquad \underline{\omega} = -\frac{1}{2}\Omega^{-1}\underline{\omega}_{\mathrm{Chr}}.$$

Lemma 1.19 (Torsion identities)**.**

$$\frac{1}{2}\mathbf{g}(\nabla_A e_3, e_4) = -\zeta_A, \qquad \eta = \zeta + \not{d}\log\Omega, \qquad \underline{\eta} = -\zeta + \not{d}\log\Omega.$$

Proof. Since $e_A\mathbf{g}(e_3, e_4) = 0$,

$$\frac{1}{2}\mathbf{g}(\nabla_A e_3, e_4) = -\frac{1}{2}\mathbf{g}(e_3, \nabla_A e_4) = -\zeta_A.$$

Using

$$e_4 = \Omega L', \qquad e_3 = \Omega \underline{L}', \qquad \nabla_{L'} L' = 0, \qquad \nabla_{\underline{L}'} \underline{L}' = 0,$$

and the symmetry of $\nabla L'$, $\nabla \underline{L}'$, we get

$$\begin{aligned}\eta_A &= \frac{1}{2}\mathbf{g}(\nabla_3 e_4, e_A) = \frac{\Omega}{2}\mathbf{g}(\nabla_3 L', e_A) = \frac{\Omega}{2}\mathbf{g}(\nabla_A L', e_3),\\ \zeta_A &= \frac{1}{2}\mathbf{g}(\nabla_A(\Omega L'), e_3) = -e_A(\log\Omega) + \frac{\Omega}{2}\mathbf{g}(\nabla_A L', e_3).\end{aligned}$$

Thus

$$\eta_A = \zeta_A + e_A(\log\Omega).$$

Similarly,

$$\begin{aligned}\underline{\eta}_A &= \frac{1}{2}\mathbf{g}(\nabla_4 e_3, e_A) = \frac{\Omega}{2}\mathbf{g}(\nabla_4 \underline{L}', e_A) = \frac{\Omega}{2}\mathbf{g}(\nabla_A \underline{L}', e_4),\\ -\zeta_A &= \frac{1}{2}\mathbf{g}(\nabla_A(\Omega \underline{L}'), e_4) = -e_A(\log\Omega) + \frac{\Omega}{2}\mathbf{g}(\nabla_A \underline{L}', e_4).\end{aligned}$$

Therefore

$$\underline{\eta}_A = -\zeta_A + e_A(\log\Omega).$$

□

Lemma 1.20 (Non-affinity)**.**

$$\nabla_4 e_4 = -2\omega e_4, \qquad \nabla_3 e_3 = -2\underline{\omega} e_3.$$

Equivalently,

$$\nabla_L L = -4\Omega\omega\, L, \qquad \nabla_{\underline{L}} \underline{L} = -4\Omega\underline{\omega}\, \underline{L}.$$

Proof. Since $e_4 = \Omega L'$, $e_3 = \Omega\underline{L}'$, and $\nabla_{L'} L' = \nabla_{\underline{L}'} \underline{L}' = 0$,

$$\nabla_4 e_4 = e_4(\log\Omega) e_4, \qquad \nabla_3 e_3 = e_3(\log\Omega) e_3.$$

On the other hand,

$$0 = e_4\mathbf{g}(e_3, e_4) = \mathbf{g}(\nabla_4 e_3, e_4) + \mathbf{g}(e_3, \nabla_4 e_4),$$

and hence

$$\mathbf{g}(\nabla_4 e_3, e_4) = 2e_4(\log\Omega).$$

Therefore

$$\omega = -\frac{1}{4}\mathbf{g}(\nabla_4 e_3, e_4) = -\frac{1}{2}e_4(\log\Omega),$$

so

$$\nabla_4 e_4 = e_4(\log\Omega) e_4 = -2\omega e_4.$$

The $\underline{\omega}$-identity is identical:

$$\nabla_3 e_3 = -2\underline{\omega} e_3.$$

Since $L = \Omega e_4$,

$$\begin{aligned}\nabla_L L &= \nabla_{\Omega e_4}(\Omega e_4) = \Omega e_4(\Omega) e_4 + \Omega^2 \nabla_4 e_4\\ &= \Omega^2 e_4(\log\Omega) e_4 - 2\Omega^2\omega e_4 = -4\Omega^2\omega e_4 = -4\Omega\omega L.\end{aligned}$$

The $\underline{L}$-identity is the conjugate one. □

1.5 Frame derivative formulas

Lemma 1.21 (Frame derivative formulas)**.** *The spacetime derivatives of the null frame are*

$$\nabla_A e_4 = \chi_A{}^B e_B - \zeta_A e_4, \qquad \nabla_A e_3 = \underline{\chi}_A{}^B e_B + \zeta_A e_3,$$
$$\nabla_A e_B = \nabla\!\!\!/_A e_B + \frac{1}{2}\chi_{AB} e_3 + \frac{1}{2}\underline{\chi}_{AB} e_4,$$
$$\nabla_4 e_4 = -2\omega e_4, \qquad \nabla_3 e_3 = -2\underline{\omega} e_3,$$
$$\nabla_4 e_3 = 2\omega e_3 + 2\underline{\eta}^\#, \qquad \nabla_3 e_4 = 2\underline{\omega} e_4 + 2\eta^\#.$$

Moreover, for S-vectors,

$$\nabla_4 e_A = (\nabla_4 e_A)^\top + \underline{\eta}_A e_4, \qquad \nabla_3 e_A = (\nabla_3 e_A)^\top + \eta_A e_3.$$

Proof. For $\nabla_A e_4$,

$$\mathbf{g}(\nabla_A e_4, e_B) = \chi_{AB}, \qquad \mathbf{g}(\nabla_A e_4, e_3) = 2\zeta_A, \qquad \mathbf{g}(\nabla_A e_4, e_4) = 0.$$

Since $\mathbf{g}(e_3, e_4) = -2$, this gives

$$\nabla_A e_4 = \chi_A{}^B e_B - \zeta_A e_4.$$

Similarly,

$$\mathbf{g}(\nabla_A e_3, e_B) = \underline{\chi}_{AB}, \qquad \mathbf{g}(\nabla_A e_3, e_4) = -2\zeta_A, \qquad \mathbf{g}(\nabla_A e_3, e_3) = 0,$$

and therefore

$$\nabla_A e_3 = \underline{\chi}_A{}^B e_B + \zeta_A e_3.$$

For $\nabla_A e_B$, the S-part is $\nabla\!\!\!/_A e_B$, while

$$\mathbf{g}(\nabla_A e_B, e_4) = -\chi_{AB}, \qquad \mathbf{g}(\nabla_A e_B, e_3) = -\underline{\chi}_{AB}.$$

Thus

$$\nabla_A e_B = \nabla\!\!\!/_A e_B + \frac{1}{2}\chi_{AB} e_3 + \frac{1}{2}\underline{\chi}_{AB} e_4.$$

The formulas for $\nabla_4 e_4$ and $\nabla_3 e_3$ are the non-affinity identities. For $\nabla_4 e_3$, the S-part is fixed by

$$\mathbf{g}(\nabla_4 e_3, e_A) = 2\underline{\eta}_A.$$

Also

$$\mathbf{g}(\nabla_4 e_3, e_4) = -4\omega, \qquad \mathbf{g}(\nabla_4 e_3, e_3) = 0.$$

Hence

$$\nabla_4 e_3 = 2\omega e_3 + 2\underline{\eta}^\#.$$

Similarly,

$$\nabla_3 e_4 = 2\underline{\omega} e_4 + 2\eta^\#.$$

Finally,

$$\mathbf{g}(\nabla_4 e_A, e_3) = -\mathbf{g}(e_A, \nabla_4 e_3) = -2\underline{\eta}_A, \qquad \mathbf{g}(\nabla_4 e_A, e_4) = 0,$$

so the normal part of $\nabla_4 e_A$ is $\underline{\eta}_A e_4$. Likewise,

$$\mathbf{g}(\nabla_3 e_A, e_4) = -\mathbf{g}(e_A, \nabla_3 e_4) = -2\eta_A, \qquad \mathbf{g}(\nabla_3 e_A, e_3) = 0,$$

so the normal part of $\nabla_3 e_A$ is $\eta_A e_3$. □

Lemma 1.22 (Torsion and non-integrability)**.** *With $L = \Omega e_4$ and $\underline{L} = \Omega e_3$,*

$$[L, \underline{L}] = -4\Omega^2 \zeta^\#.$$

Equivalently,

$$[\underline{L}, L] = 4\Omega^2 \zeta^\#.$$

Proof. Using $L = \Omega e_4$, $\underline{L} = \Omega e_3$,

$$\begin{aligned}[L, \underline{L}] &= \nabla_L \underline{L} - \nabla_{\underline{L}} L \\ &= \Omega e_4(\Omega) e_3 + \Omega^2 \nabla_4 e_3 - \Omega e_3(\Omega) e_4 - \Omega^2 \nabla_3 e_4 \\ &= \Omega^2 \big(e_4(\log \Omega) e_3 + 2\omega e_3 + 2\underline{\eta}^\# \\ &\qquad - e_3(\log \Omega) e_4 - 2\underline{\omega} e_4 - 2\eta^\# \big).\end{aligned}$$

Since

$$e_4(\log \Omega) = -2\omega, \qquad e_3(\log \Omega) = -2\underline{\omega},$$

the e_3, e_4-terms cancel and

$$[L, \underline{L}] = 2\Omega^2 (\underline{\eta}^\# - \eta^\#).$$

Using

$$\eta = \zeta + \nabla\!\!\!/ \log \Omega, \qquad \underline{\eta} = -\zeta + \nabla\!\!\!/ \log \Omega,$$

we obtain

$$[L, \underline{L}] = -4\Omega^2 \zeta^\#.$$

□

1.6 Canonical coordinates

Fix a section $S_{u_0, \underline{u}_0}$. The canonical coordinates are constructed as follows.

1. Fix the initial bifurcate sphere and the two initial null hypersurfaces

   $$S_{u_0, \underline{u}_0}, \qquad C_{u_0}, \qquad \underline{C}_{\underline{u}_0},$$

   where

   $$C_{u_0} = \{u = u_0\}, \qquad \underline{C}_{\underline{u}_0} = \{\underline{u} = \underline{u}_0\}, \qquad S_{u_0, \underline{u}_0} = C_{u_0} \cap \underline{C}_{\underline{u}_0}.$$

2. Choose local angular coordinates

   $$(\vartheta^1, \vartheta^2)$$

   on $S_{u_0, \underline{u}_0}$.

3. Extend ϑ^A first along C_{u_0} by the outgoing null flow:

   $$L(\vartheta^A) = 0 \qquad \text{on } C_{u_0}.$$

   Thus on C_{u_0},

   $$L = \partial_{\underline{u}}.$$

4. Extend ϑ^A from C_{u_0} to the spacetime region by the incoming null flow:

   $$\underline{L}(\vartheta^A) = 0.$$

   Since

   $$\underline{L} u = 1, \qquad \underline{L}\underline{u} = 0, \qquad \underline{L} \vartheta^A = 0,$$

   one has

   $$\underline{L} = \partial_u.$$

5. Define the angular shift b^A by

$$b^A := L(\vartheta^A).$$

Since

$$Lu = 0, \qquad L\underline{u} = 1, \qquad L\vartheta^A = b^A,$$

one has

$$L = \partial_{\underline{u}} + b^A \partial_A.$$

The normalization on the initial outgoing hypersurface gives

$$b^A = 0 \qquad \text{on } C_{u_0}.$$

6. Set

$$\theta^A := d\vartheta^A - b^A d\underline{u}.$$

Then

$$\theta^A(L) = d\vartheta^A(L) - b^A d\underline{u}(L) = b^A - b^A = 0,$$
$$\theta^A(\underline{L}) = d\vartheta^A(\underline{L}) - b^A d\underline{u}(\underline{L}) = 0,$$

and

$$\theta^A(\partial_B) = \delta^A_B.$$

7. Since

$$d\vartheta^A = \theta^A + b^A d\underline{u},$$

the change from

$$du, \qquad d\underline{u}, \qquad d\vartheta^1, \qquad d\vartheta^2$$

to

$$du, \qquad d\underline{u}, \qquad \theta^1, \qquad \theta^2$$

is triangular with determinant 1. Hence

$$du, \qquad d\underline{u}, \qquad \theta^1, \qquad \theta^2$$

is a coframe.

8. In the dual frame

$$L = \partial_{\underline{u}} + b^A \partial_A, \qquad \underline{L} = \partial_u, \qquad \partial_A,$$

the metric satisfies

$$\mathbf{g}(L, L) = 0, \qquad \mathbf{g}(\underline{L}, \underline{L}) = 0, \qquad \mathbf{g}(L, \partial_A) = \mathbf{g}(\underline{L}, \partial_A) = 0,$$
$$\mathbf{g}(L, \underline{L}) = -2\Omega^2, \qquad \mathbf{g}(\partial_A, \partial_B) = \not{g}_{AB}.$$

Therefore

$$\mathbf{g} = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \not{g}_{AB}\theta^A \otimes \theta^B,$$

that is,

$$\mathbf{g} = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \not{g}_{AB}(d\vartheta^A - b^A d\underline{u}) \otimes (d\vartheta^B - b^B d\underline{u}).$$

9. The coordinate variables are

$$\Omega, \qquad b = b^A \partial_A, \qquad \not{g} = \not{g}_{AB} d\vartheta^A d\vartheta^B.$$

10. The shift is determined by the torsion:

$$[L, \underline{L}] = -4\Omega^2 \zeta^{\#}.$$

Since

$$[L, \underline{L}] = [\partial_{\underline{u}} + b^A \partial_A, \partial_u] = -(\partial_u b^A)\partial_A,$$

one obtains

$$\partial_u b^A = 4\Omega^2 \zeta^A, \qquad b^A|_{C_{u_0}} = 0.$$

Remark 1.23 (Shift and torsion). The shift b records the non-commutativity of the two null flows. Indeed,

$$[L, \underline{L}] = [\partial_{\underline{u}} + b^A \partial_A, \partial_u] = -(\partial_u b^A)\partial_A.$$

Together with

$$[L, \underline{L}] = -4\Omega^2 \zeta^{\#},$$

this gives

$$\partial_u b^A = 4\Omega^2 \zeta^A.$$

Thus the u-variation of the angular shift is governed by the torsion.

1.7 Curvature convention and Weyl components

We use Christodoulou's curvature convention

$$\mathbf{Rm}(W, Z, X, Y) := \mathbf{g}\big(W, \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\big).$$

Thus

$$\mathbf{Rm}(W, Z, X, Y) = \mathbf{g}(W, R(X, Y)Z).$$

In dimension 4,

$$\begin{aligned}\mathbf{Rm}_{\alpha\beta\gamma\delta} = \mathbf{W}_{\alpha\beta\gamma\delta} &+ \frac{1}{2}\big(\mathbf{g}_{\alpha\gamma}\mathbf{Ric}_{\beta\delta} + \mathbf{g}_{\beta\delta}\mathbf{Ric}_{\alpha\gamma} - \mathbf{g}_{\alpha\delta}\mathbf{Ric}_{\beta\gamma} - \mathbf{g}_{\beta\gamma}\mathbf{Ric}_{\alpha\delta}\big) \\ &- \frac{\mathbf{R}}{6}\big(\mathbf{g}_{\alpha\gamma}\mathbf{g}_{\beta\delta} - \mathbf{g}_{\alpha\delta}\mathbf{g}_{\beta\gamma}\big).\end{aligned}$$

With respect to the normalized null frame

$$\mathbf{g}(e_3, e_4) = -2,$$

define

$$\begin{aligned} \alpha_{AB} &:= \mathbf{W}_{A4B4}, & \underline{\alpha}_{AB} &:= \mathbf{W}_{A3B3}, \\ \beta_A &:= \frac{1}{2}\mathbf{W}_{A434}, & \underline{\beta}_A &:= \frac{1}{2}\mathbf{W}_{A334}, \\ \rho &:= \frac{1}{4}\mathbf{W}_{3434}, & \sigma &:= \frac{1}{4}\not{\epsilon}^{AB}\mathbf{W}_{AB34}. \end{aligned}$$

Hence

$$\mathbf{W}_{A434} = 2\beta_A, \qquad \mathbf{W}_{A334} = 2\underline{\beta}_A, \qquad \mathbf{W}_{3434} = 4\rho, \qquad \mathbf{W}_{AB34} = 2\sigma\not{\epsilon}_{AB}.$$

Lemma 1.24 (Algebraic identities for Weyl components)**.**

$$\alpha_{AB} = \alpha_{BA}, \qquad \underline{\alpha}_{AB} = \underline{\alpha}_{BA}, \qquad \not{g}^{AB}\alpha_{AB} = 0, \qquad \not{g}^{AB}\underline{\alpha}_{AB} = 0.$$

Moreover,

$$\mathbf{W}_{A4B3} = -\rho\not{g}_{AB} - \sigma\not{\epsilon}_{AB}.$$

Consequently, $\mathbf{W}$ *is encoded by*

$$\alpha, \qquad \beta, \qquad \rho, \qquad \sigma, \qquad \underline{\beta}, \qquad \underline{\alpha}.$$

Proof. The identities for $\alpha, \underline{\alpha}$ follow from the pair symmetry and trace-freeness of $\mathbf{W}$. For $\mathbf{W}_{A4B3}$,

$$\begin{aligned} 0 &= \mathbf{g}^{\mu\nu}\mathbf{W}_{\mu 4 \nu 3} \\ &= \slashed{g}^{AB}\mathbf{W}_{A4B3} - \frac{1}{2}\mathbf{W}_{3443} - \frac{1}{2}\mathbf{W}_{4433} \\ &= \slashed{g}^{AB}\mathbf{W}_{A4B3} + 2\rho. \end{aligned}$$

Thus

$$\slashed{g}^{AB}\mathbf{W}_{A4B3} = -2\rho.$$

The algebraic Bianchi identity gives

$$\mathbf{W}_{A4B3} + \mathbf{W}_{AB34} + \mathbf{W}_{A34B} = 0.$$

Since

$$\mathbf{W}_{A34B} = \mathbf{W}_{4BA3} = -\mathbf{W}_{B4A3},$$

we get

$$\mathbf{W}_{A4B3} - \mathbf{W}_{B4A3} = -\mathbf{W}_{AB34} = -2\sigma\slashed{\epsilon}_{AB}.$$

The symmetric trace-free part of $\mathbf{W}_{A4B3}$ vanishes by the algebraic Weyl identities in dimension 4. Hence

$$\mathbf{W}_{A4B3} = -\rho\slashed{g}_{AB} - \sigma\slashed{\epsilon}_{AB}.$$

□

The curvature decompositions used in the null structure equations are

$$\mathbf{Rm}_{A434} = 2\beta_A + \mathbf{Ric}_{4A}, \qquad \mathbf{Rm}_{A334} = 2\underline{\beta}_A - \mathbf{Ric}_{3A},$$

and

$$\frac{1}{2}\slashed{\epsilon}^{AB}\slashed{\epsilon}^{CD}\mathbf{Rm}_{ABCD} = -\rho + \frac{1}{2}\mathrm{tr}_S\mathbf{Ric} - \frac{1}{6}\mathbf{R}.$$

Indeed,

$$\begin{aligned} \mathbf{Rm}_{A434} &= \mathbf{W}_{A434} + \frac{1}{2}\big(\mathbf{g}_{A3}\mathbf{Ric}_{44} + \mathbf{g}_{44}\mathbf{Ric}_{A3} - \mathbf{g}_{A4}\mathbf{Ric}_{43} - \mathbf{g}_{43}\mathbf{Ric}_{A4}\big) \\ &\quad - \frac{\mathbf{R}}{6}\big(\mathbf{g}_{A3}\mathbf{g}_{44} - \mathbf{g}_{A4}\mathbf{g}_{43}\big) \\ &= \mathbf{W}_{A434} + \mathbf{Ric}_{A4} = 2\beta_A + \mathbf{Ric}_{4A}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{Rm}_{A334} &= \mathbf{W}_{A334} + \frac{1}{2}\big(\mathbf{g}_{A3}\mathbf{Ric}_{34} + \mathbf{g}_{34}\mathbf{Ric}_{A3} - \mathbf{g}_{A4}\mathbf{Ric}_{33} - \mathbf{g}_{33}\mathbf{Ric}_{A4}\big) \\ &\quad - \frac{\mathbf{R}}{6}\big(\mathbf{g}_{A3}\mathbf{g}_{34} - \mathbf{g}_{A4}\mathbf{g}_{33}\big) \\ &= \mathbf{W}_{A334} - \mathbf{Ric}_{A3} = 2\underline{\beta}_A - \mathbf{Ric}_{3A}. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{1}{2}\slashed{\epsilon}^{AB}\slashed{\epsilon}^{CD}\mathbf{Rm}_{ABCD} &= \frac{1}{2}\slashed{\epsilon}^{AB}\slashed{\epsilon}^{CD}\mathbf{W}_{ABCD} \\ &\quad + \frac{1}{4}\slashed{\epsilon}^{AB}\slashed{\epsilon}^{CD}\big(\slashed{g}_{AC}\mathbf{Ric}_{BD} + \slashed{g}_{BD}\mathbf{Ric}_{AC} - \slashed{g}_{AD}\mathbf{Ric}_{BC} - \slashed{g}_{BC}\mathbf{Ric}_{AD}\big) \\ &\quad - \frac{\mathbf{R}}{12}\slashed{\epsilon}^{AB}\slashed{\epsilon}^{CD}\big(\slashed{g}_{AC}\slashed{g}_{BD} - \slashed{g}_{AD}\slashed{g}_{BC}\big) \\ &= -\rho + \frac{1}{2}\mathrm{tr}_S\mathbf{Ric} - \frac{1}{6}\mathbf{R}. \end{aligned}$$

Remark 1.25. In vacuum,

$$\mathbf{Rm} = \mathbf{W}.$$

In a general spacetime, the null structure equations contain the additional Ricci curvature terms obtained from the above decomposition of $\mathbf{Rm}$ into $\mathbf{W}, \mathbf{Ric}, \mathbf{R}$.

1.8 Examples

Example 1.26 (Minkowski spacetime)**.** Let

$$\mathbf{g} = -dt^2 + dr^2 + r^2 \gamma_{\mathbb{S}^2}$$

on $\mathbb{R}^{1+3}$, and set

$$u = \frac{1}{2}(t-r), \qquad \underline{u} = \frac{1}{2}(t+r).$$

Then

$$t = u + \underline{u}, \qquad r = \underline{u} - u,$$

and

$$\mathbf{g} = -4\, du\, d\underline{u} + r^2 \gamma_{\mathbb{S}^2}.$$

Thus

$$\Omega = 1, \qquad L' = e_4 = L = \partial_t + \partial_r = \partial_{\underline{u}},$$

and

$$\underline{L}' = e_3 = \underline{L} = \partial_t - \partial_r = \partial_u.$$

The sections

$$S_{u,\underline{u}}$$

are the round spheres of radius

$$r = \underline{u} - u,$$

with induced metric

$$\not{g} = r^2 \gamma_{\mathbb{S}^2}.$$

Since

$$e_4(r) = 1, \qquad e_3(r) = -1,$$

we get

$$\chi_{AB} = \frac{1}{r} \not{g}_{AB}, \qquad \mathrm{tr}\chi = \frac{2}{r},$$

and

$$\underline{\chi}_{AB} = -\frac{1}{r} \not{g}_{AB}, \qquad \mathrm{tr}\underline{\chi} = -\frac{2}{r}.$$

Hence

$$\hat{\chi} = 0, \qquad \underline{\hat{\chi}} = 0.$$

Moreover,

$$\zeta = 0, \qquad \eta = 0, \qquad \underline{\eta} = 0, \qquad \omega = 0, \qquad \underline{\omega} = 0.$$

The curvature vanishes:

$$\alpha = \beta = \rho = \sigma = \underline{\beta} = \underline{\alpha} = 0.$$

The Gauss equation becomes

$$K = -\frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) = -\frac{1}{4}\frac{2}{r}\left(-\frac{2}{r}\right) = \frac{1}{r^2},$$

as expected for the round sphere of radius r.

Example 1.27 (Schwarzschild exterior)**.** Let

$$\mathbf{g} = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\gamma_{\mathbb{S}^2}$$

on the exterior region $r > 2M$. Set

$$\Upsilon := 1 - \frac{2M}{r}, \qquad \frac{dr_*}{dr} = \Upsilon^{-1},$$

and define

$$u = \frac{1}{2}(t - r_*), \qquad \underline{u} = \frac{1}{2}(t + r_*).$$

Then

$$t = u + \underline{u}, \qquad r_* = \underline{u} - u,$$

and

$$\mathbf{g} = -4\Upsilon\, du\, d\underline{u} + r^2\gamma_{\mathbb{S}^2}.$$

Thus

$$\Omega^2 = \Upsilon, \qquad L = \partial_{\underline{u}}, \qquad \underline{L} = \partial_u,$$

while

$$L' = \Omega^{-2}\partial_{\underline{u}}, \qquad \underline{L}' = \Omega^{-2}\partial_u,$$

and

$$e_4 = \Omega^{-1}\partial_{\underline{u}}, \qquad e_3 = \Omega^{-1}\partial_u.$$

Since

$$\partial_{\underline{u}} r = \Omega^2, \qquad \partial_u r = -\Omega^2,$$

we have

$$e_4(r) = \Omega, \qquad e_3(r) = -\Omega.$$

Therefore

$$\chi_{AB} = \frac{\Omega}{r}\not{g}_{AB}, \qquad \mathrm{tr}\chi = \frac{2\Omega}{r},$$

and

$$\underline{\chi}_{AB} = -\frac{\Omega}{r}\not{g}_{AB}, \qquad \mathrm{tr}\underline{\chi} = -\frac{2\Omega}{r}.$$

Again

$$\hat{\chi} = 0, \qquad \underline{\hat{\chi}} = 0,$$

because the sections are round spheres. Also

$$\zeta = 0, \qquad \eta = 0, \qquad \underline{\eta} = 0.$$

The lapse coefficients are

$$e_4(\log\Omega) = \frac{M}{r^2\Omega}, \qquad e_3(\log\Omega) = -\frac{M}{r^2\Omega},$$

hence

$$\omega = -\frac{M}{2r^2\Omega}, \qquad \underline{\omega} = \frac{M}{2r^2\Omega}.$$

The only nonzero Weyl component is the Coulomb component

$$\rho = -\frac{2M}{r^3}.$$

Indeed, since

$$K = \frac{1}{r^2}, \qquad \mathrm{tr}\chi = \frac{2\Omega}{r}, \qquad \mathrm{tr}\underline{\chi} = -\frac{2\Omega}{r}, \qquad \hat{\chi} = \underline{\hat{\chi}} = 0,$$

the vacuum Gauss equation gives

$$\frac{1}{r^2} = -\rho - \frac{1}{4}\frac{2\Omega}{r}\left(-\frac{2\Omega}{r}\right) = -\rho + \frac{\Omega^2}{r^2}.$$

Thus

$$\rho = \frac{\Omega^2 - 1}{r^2} = -\frac{2M}{r^3}.$$

For $M = 0$, this reduces to the Minkowski example.

1.9 Model comparison: Cauchy and characteristic data for the wave equation

We use the scalar wave equation on Minkowski spacetime as a model for the difference between the spacelike Cauchy formulation and the double-null characteristic formulation. In rectangular coordinates,

$$\mathbf{g} = -dt^2 + \sum_{i=1}^{3}(dx^i)^2, \qquad \Box_{\mathbf{g}}\phi = 0,$$

that is,

$$-\partial_t^2\phi + \sum_{i=1}^{3}\partial_i^2\phi = 0.$$

1.9.1 Cauchy problem: second-order form

Let

$$\Sigma_0 = \{t = 0\}.$$

The Cauchy problem treats the wave equation as a second-order evolution equation in the transverse variable t:

$$\partial_t^2\phi = \sum_{i=1}^{3}\partial_i^2\phi.$$

Thus the initial data are

$$\phi|_{\Sigma_0} = \phi_0, \qquad \partial_t\phi|_{\Sigma_0} = \phi_1.$$

The first datum is the restriction of the field to Σ_0. The second datum is the transverse derivative.

This is the scalar model for the $3+1$ Cauchy formulation of the Einstein equations: one prescribes geometric data on a spacelike hypersurface together with one transverse time derivative, subject to constraints.

1.9.2 Characteristic problem: first-order system for second derivatives

Now introduce double-null coordinates

$$u = \frac{1}{2}(t - r), \qquad \underline{u} = \frac{1}{2}(t + r), \qquad r = \underline{u} - u.$$

Then

$$\mathbf{g} = -4\,du\,d\underline{u} + r^2\gamma_{\mathbb{S}^2}.$$

In the double-null notation,

$$\Omega = 1, \qquad L = D = \partial_{\underline{u}}, \qquad \underline{L} = \underline{D} = \partial_u, \qquad \not{g} = r^2\gamma_{\mathbb{S}^2}.$$

The wave equation becomes

$$-D\underline{D}\phi + \frac{1}{r}(D\phi - \underline{D}\phi) + \Delta_{\not{g}}\phi = 0.$$

Thus

$$D\underline{D}\phi = \mathrm{d\!/\!iv}\, \nabla\!\!\!/ \phi + \frac{1}{r}(D\phi - \underline{D}\phi).$$

This should be viewed as the scalar analogue of a null structure equation: it determines the mixed second derivative from the tangential second derivatives and lower-order terms.

The analogue of the curvature variables is the package of second derivatives

$$D^2\phi, \qquad D\nabla\!\!\!/\phi, \qquad \nabla\!\!\!/^2\phi, \qquad \underline{D}\nabla\!\!\!/\phi, \qquad \underline{D}^2\phi.$$

The wave equation gives a first-order double-null system for this package. Along C_u, the transverse second derivatives are transported by

$$D(\underline{D}\nabla\!\!\!/\phi) = \nabla\!\!\!/ \left(\mathrm{d\!/\!iv}\, \nabla\!\!\!/ \phi + \frac{1}{r}(D\phi - \underline{D}\phi) \right),$$

and

$$D(\underline{D}^2\phi) = \underline{D} \left(\mathrm{d\!/\!iv}\, \nabla\!\!\!/ \phi + \frac{1}{r}(D\phi - \underline{D}\phi) \right).$$

Along $\underline{C}_{\underline{u}}$, the conjugate transverse second derivatives are transported by

$$\underline{D}(D\nabla\!\!\!/\phi) = \nabla\!\!\!/ \left(\mathrm{d\!/\!iv}\, \nabla\!\!\!/ \phi + \frac{1}{r}(D\phi - \underline{D}\phi) \right),$$

and

$$\underline{D}(D^2\phi) = D \left(\mathrm{d\!/\!iv}\, \nabla\!\!\!/ \phi + \frac{1}{r}(D\phi - \underline{D}\phi) \right).$$

The angular part is governed by the intrinsic identities

$$D(\nabla\!\!\!/^2\phi) = \nabla\!\!\!/^2(D\phi) + \text{lower-order terms}, \qquad \underline{D}(\nabla\!\!\!/^2\phi) = \nabla\!\!\!/^2(\underline{D}\phi) + \text{lower-order terms}.$$

Thus the characteristic formulation is not naturally a second-order evolution equation for ϕ. It is naturally a first-order system for the null and angular second derivatives of ϕ, with the wave equation providing the mixed component $D\underline{D}\phi$.

Let

$$C_{u_0} = \{u = u_0\}, \qquad \underline{C}_{\underline{u}_0} = \{\underline{u} = \underline{u}_0\}, \qquad S_{u_0,\underline{u}_0} = C_{u_0} \cap \underline{C}_{\underline{u}_0}.$$

The characteristic data in second-order scalar form are simply

$$\phi|_{C_{u_0}} = \phi_+, \qquad \phi|_{\underline{C}_{\underline{u}_0}} = \phi_-,$$

with the corner compatibility condition

$$\phi_+|_{S_{u_0,\underline{u}_0}} = \phi_-|_{S_{u_0,\underline{u}_0}}.$$

On C_{u_0}, the tangential derivatives

$$D\phi, \qquad \nabla\!\!\!/\phi, \qquad D^2\phi, \qquad D\nabla\!\!\!/\phi, \qquad \nabla\!\!\!/^2\phi$$

are determined by ϕ_+. The missing transverse derivatives

$$\underline{D}\phi, \qquad \underline{D}\nabla\!\!\!/\phi, \qquad \underline{D}^2\phi$$

are not freely prescribed along C_{u_0}; they are obtained from the corner values supplied by ϕ_- and then transported along C_{u_0} by the above equations.

Similarly, on $\underline{C}_{\underline{u}_0}$, the tangential derivatives

$$\underline{D}\phi, \qquad \nabla\!\!\!/\phi, \qquad \underline{D}^2\phi, \qquad \underline{D}\nabla\!\!\!/\phi, \qquad \nabla\!\!\!/^2\phi$$

are determined by ϕ_-. The missing transverse derivatives

$$D\phi, \qquad D\nabla\!\!\!/\phi, \qquad D^2\phi$$

are obtained from the corner values supplied by ϕ_+ and then transported along $\underline{C}_{\underline{u}_0}$.

Remark 1.28. The comparison is:

$$\text{Cauchy problem} \quad \Longleftrightarrow \quad \text{second-order evolution in a spacelike time direction,}$$

whereas

$$\text{characteristic problem} \quad \Longleftrightarrow \quad \text{first-order transport system for null curvature quantities.}$$

For the scalar wave equation, the curvature quantities are the second derivatives of ϕ. For the Einstein equations, the corresponding quantities are the null curvature components

$$\alpha, \qquad \beta, \qquad \rho, \qquad \sigma, \qquad \underline{\beta}, \qquad \underline{\alpha},$$

and the first-order characteristic system is the null Bianchi system.

2 The null structure equations

We state the null structure equations without imposing $\mathbf{Ric} = 0$. Write

$$\mathbf{Ric}_{\mu\nu} = \mathbf{Ric}(e_\mu, e_\nu), \qquad \mathbf{R} = \mathrm{tr}_{\mathbf{g}}\mathbf{Ric}, \qquad \mathrm{tr}_S\mathbf{Ric} = \not{g}^{AB}\mathbf{Ric}_{AB}.$$

The curvature decomposition is

$$\begin{aligned}\mathbf{Rm}_{\alpha\beta\gamma\delta} = \mathbf{W}_{\alpha\beta\gamma\delta} &+ \frac{1}{2}\left(\mathbf{g}_{\alpha\gamma}\mathbf{Ric}_{\beta\delta} + \mathbf{g}_{\beta\delta}\mathbf{Ric}_{\alpha\gamma} - \mathbf{g}_{\alpha\delta}\mathbf{Ric}_{\beta\gamma} - \mathbf{g}_{\beta\gamma}\mathbf{Ric}_{\alpha\delta}\right)\\ &- \frac{\mathbf{R}}{6}\left(\mathbf{g}_{\alpha\gamma}\mathbf{g}_{\beta\delta} - \mathbf{g}_{\alpha\delta}\mathbf{g}_{\beta\gamma}\right).\end{aligned}$$

For symmetric S-two-tensors θ, θ', set

$$(\theta \times \theta')_{AB} = \theta_A{}^C\theta'_{CB}, \qquad (\theta, \theta') = \theta_{AB}\theta'^{AB}, \qquad \theta \wedge \theta' = \not{\epsilon}^{AB}\theta_A{}^C\theta'_{BC}.$$

2.1 Raychaudhuri equations and focusing

Proposition 2.1 (Raychaudhuri equations)**.** *The null second fundamental forms satisfy*

$$D\chi = -2\Omega\omega\,\chi + \Omega\chi \times \chi - \Omega\alpha - \frac{\Omega}{2}\mathbf{Ric}_{44}\not{g},$$

and

$$\underline{D}\underline{\chi} = -2\Omega\underline{\omega}\,\underline{\chi} + \Omega\underline{\chi} \times \underline{\chi} - \Omega\underline{\alpha} - \frac{\Omega}{2}\mathbf{Ric}_{33}\not{g}.$$

Taking traces gives

$$D\mathrm{tr}\chi = -\frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \Omega|\hat{\chi}|^2 - 2\Omega\omega\,\mathrm{tr}\chi - \Omega\mathbf{Ric}_{44},$$

and

$$\underline{D}\mathrm{tr}\underline{\chi} = -\frac{\Omega}{2}(\mathrm{tr}\underline{\chi})^2 - \Omega|\hat{\underline{\chi}}|^2 - 2\Omega\underline{\omega}\,\mathrm{tr}\underline{\chi} - \Omega\mathbf{Ric}_{33}.$$

Proof. Extend $X, Y \in T_pS_{u,\underline{u}}$ along the L-generator by

$$[L, X] = [L, Y] = 0.$$

Then, using Christodoulou's convention

$$\mathbf{Rm}(W, Z, X, Y) = \mathbf{g}(W, R(X, Y)Z),$$

we compute

$$\begin{aligned}
D\chi(X,Y) &= L\mathbf{g}(\nabla_X e_4, Y) \\
&= \mathbf{g}(\nabla_L \nabla_X e_4, Y) + \mathbf{g}(\nabla_X e_4, \nabla_L Y) \\
&= \mathbf{g}(\nabla_X \nabla_L e_4, Y) + \mathbf{Rm}(Y, e_4, L, X) + \mathbf{g}(\nabla_X e_4, \nabla_Y L) \\
&= -2\Omega\omega\,\chi(X,Y) + \Omega(\chi \times \chi)(X,Y) - \Omega\mathbf{Rm}(X, e_4, Y, e_4).
\end{aligned}$$

Indeed,

$$\nabla_L e_4 = \Omega \nabla_4 e_4 = -2\Omega\omega e_4,$$

so

$$\mathbf{g}(\nabla_X \nabla_L e_4, Y) = -2\Omega\omega\,\chi(X,Y).$$

Also,

$$\nabla_Y L = Y(\Omega) e_4 + \Omega \nabla_Y e_4,$$

and therefore

$$\mathbf{g}(\nabla_X e_4, \nabla_Y L) = \Omega(\chi \times \chi)(X,Y).$$

Finally,

$$\mathbf{Rm}(Y, e_4, L, X) = \Omega\mathbf{Rm}(Y, e_4, e_4, X) = -\Omega\mathbf{Rm}(X, e_4, Y, e_4),$$

by the pair symmetry and antisymmetry of $\mathbf{Rm}$.

For the curvature term,

$$\begin{aligned}
\mathbf{Rm}_{A4B4} &= \mathbf{W}_{A4B4} + \frac{1}{2}\left(\not{g}_{AB}\mathbf{Ric}_{44} + \mathbf{g}_{44}\mathbf{Ric}_{AB} - \mathbf{g}_{A4}\mathbf{Ric}_{4B} - \mathbf{g}_{4B}\mathbf{Ric}_{A4} \right) \\
&\quad - \frac{\mathbf{R}}{6}\left(\not{g}_{AB}\mathbf{g}_{44} - \mathbf{g}_{A4}\mathbf{g}_{4B} \right) \\
&= \alpha_{AB} + \frac{1}{2}\mathbf{Ric}_{44}\not{g}_{AB}.
\end{aligned}$$

Thus

$$D\chi = -2\Omega\omega\,\chi + \Omega\chi \times \chi - \Omega\alpha - \frac{\Omega}{2}\mathbf{Ric}_{44}\not{g}.$$

The incoming computation is identical:

$$\begin{aligned}
\underline{D}\underline{\chi}(X,Y) &= \mathbf{g}(\nabla_{\underline{L}} \nabla_X e_3, Y) + \mathbf{g}(\nabla_X e_3, \nabla_{\underline{L}} Y) \\
&= \mathbf{g}(\nabla_X \nabla_{\underline{L}} e_3, Y) + \mathbf{Rm}(Y, e_3, \underline{L}, X) + \mathbf{g}(\nabla_X e_3, \nabla_Y \underline{L}) \\
&= -2\Omega\underline{\omega}\,\underline{\chi}(X,Y) + \Omega(\underline{\chi} \times \underline{\chi})(X,Y) - \Omega\mathbf{Rm}(X, e_3, Y, e_3).
\end{aligned}$$

Since

$$\mathbf{Rm}_{A3B3} = \underline{\alpha}_{AB} + \frac{1}{2}\mathbf{Ric}_{33}\not{g}_{AB},$$

we get

$$\underline{D}\underline{\chi} = -2\Omega\underline{\omega}\,\underline{\chi} + \Omega\underline{\chi} \times \underline{\chi} - \Omega\underline{\alpha} - \frac{\Omega}{2}\mathbf{Ric}_{33}\not{g}.$$

Taking traces requires differentiating the inverse S-metric. Since

$$D\not{g} = 2\Omega\chi, \qquad D\not{g}^{AB} = -2\Omega\chi^{AB},$$

we have, for any S-two-tensor θ,

$$D(\mathrm{tr}\theta) = \mathrm{tr}(D\theta) - 2\Omega(\chi, \theta).$$

Thus

$$D\mathrm{tr}\chi = \mathrm{tr}(D\chi) - 2\Omega|\chi|^2.$$

Using

$$\mathrm{tr}(\chi \times \chi) = |\chi|^2, \qquad \mathrm{tr}\alpha = 0, \qquad \mathrm{tr}\not{g} = 2,$$

we obtain

$$\begin{aligned} D\mathrm{tr}\chi &= -2\Omega\omega\,\mathrm{tr}\chi + \Omega|\chi|^2 - \Omega\mathbf{Ric}_{44} - 2\Omega|\chi|^2 \\ &= -2\Omega\omega\,\mathrm{tr}\chi - \Omega|\chi|^2 - \Omega\mathbf{Ric}_{44} \\ &= -\frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \Omega|\hat{\chi}|^2 - 2\Omega\omega\,\mathrm{tr}\chi - \Omega\mathbf{Ric}_{44}. \end{aligned}$$

Similarly,

$$\underline{D}\mathrm{tr}\underline{\chi} = \mathrm{tr}(\underline{D}\underline{\chi}) - 2\Omega|\underline{\chi}|^2,$$

and hence

$$\underline{D}\mathrm{tr}\underline{\chi} = -\frac{\Omega}{2}(\mathrm{tr}\underline{\chi})^2 - \Omega|\hat{\underline{\chi}}|^2 - 2\Omega\underline{\omega}\,\mathrm{tr}\underline{\chi} - \Omega\mathbf{Ric}_{33}.$$

□

Remark 2.2. These are the null analogues of the second variation equation for a hypersurface. The trace equations have the same focusing term as the Jacobi operator:

$$-\Delta_\Sigma - |A|^2 - \mathbf{Ric}(\nu,\nu).$$

Indeed, in the outgoing null direction,

$$D\mathrm{tr}\chi = -\Omega|\chi|^2 - 2\Omega\omega\,\mathrm{tr}\chi - \Omega\mathbf{Ric}_{44},$$

or equivalently

$$D\mathrm{tr}\chi = -\frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \Omega|\hat{\chi}|^2 - 2\Omega\omega\,\mathrm{tr}\chi - \Omega\mathbf{Ric}_{44}.$$

Thus the null analogue of $|A|^2 + \mathbf{Ric}(\nu,\nu)$ is

$$|\chi|^2 + \mathbf{Ric}_{44} = \frac{1}{2}(\mathrm{tr}\chi)^2 + |\hat{\chi}|^2 + \mathbf{Ric}_{44}.$$

Similarly,

$$|\underline{\chi}|^2 + \mathbf{Ric}_{33} = \frac{1}{2}(\mathrm{tr}\underline{\chi})^2 + |\hat{\underline{\chi}}|^2 + \mathbf{Ric}_{33}.$$

The Weyl components $\alpha, \underline{\alpha}$ enter the trace-free equations for the shears, but drop out of the trace equations because

$$\mathrm{tr}\alpha = \mathrm{tr}\underline{\alpha} = 0.$$

Definition 2.3 (Future trapped surface)**.** A closed spacelike two-surface S is future trapped if, with respect to its two future-directed affine null normals L' and $\underline{L}'$,

$$\mathrm{tr}\chi' < 0, \qquad \mathrm{tr}\underline{\chi}' < 0 \qquad \text{on } S.$$

Theorem 2.4 (Focusing and Penrose mechanism)**.** *Assume null energy condition (NEC)*

$$\mathbf{Ric}(V,V) \geq 0 \qquad \textit{for every null vector } V.$$

Let L' be an affine future null generator orthogonal to S. If

$$\mathrm{tr}\chi'(0) < 0,$$

then $\mathrm{tr}\chi'$ blows up to $-\infty$ before affine time

$$\frac{2}{|\mathrm{tr}\chi'(0)|}.$$

Thus a future trapped surface forces both future null congruences to focus in finite affine time. In Penrose's global setting, namely global hyperbolicity, a noncompact Cauchy hypersurface, the null convergence condition, and a closed future trapped surface, the spacetime is future null geodesically incomplete.

Proof. Since $e_4 = \Omega L'$ and L' is affine,

$$D = \Omega^2 L', \qquad \chi = \Omega\chi', \qquad \mathrm{tr}\chi = \Omega \mathrm{tr}\chi', \qquad \hat{\chi} = \Omega\hat{\chi}', \qquad \mathbf{Ric}_{44} = \Omega^2 \mathbf{Ric}(L', L'),$$

and

$$\nabla_4 e_4 = \nabla_{\Omega L'}(\Omega L') = L'(\Omega) e_4 = -2\omega e_4, \qquad \omega = -\frac{1}{2} L'(\Omega).$$

Substituting these identities into

$$D\mathrm{tr}\chi = -\frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \Omega|\hat{\chi}|^2 - 2\Omega\omega\, \mathrm{tr}\chi - \Omega\mathbf{Ric}_{44},$$

gives

$$\begin{aligned}\Omega^2 L'(\Omega\mathrm{tr}\chi') &= -\frac{\Omega^3}{2}(\mathrm{tr}\chi')^2 - \Omega^3|\hat{\chi}'|^2 + \Omega^2 L'(\Omega)\mathrm{tr}\chi' - \Omega^3\mathbf{Ric}(L', L') \\ &= \Omega^2 L'(\Omega)\mathrm{tr}\chi' + \Omega^3 L'(\mathrm{tr}\chi').\end{aligned}$$

Canceling the common term and dividing by Ω^3,

$$L'(\mathrm{tr}\chi') = -\frac{1}{2}(\mathrm{tr}\chi')^2 - |\hat{\chi}'|^2 - \mathbf{Ric}(L', L').$$

Thus, if $\mathbf{Ric}(V, V) \geq 0$ for every null V,

$$L'(\mathrm{tr}\chi') \leq -\frac{1}{2}(\mathrm{tr}\chi')^2.$$

Let s be the affine parameter. If $\mathrm{tr}\chi'(0) < 0$, then

$$\frac{d}{ds}\frac{1}{\mathrm{tr}\chi'} = -\frac{L'(\mathrm{tr}\chi')}{(\mathrm{tr}\chi')^2} \geq \frac{1}{2}, \qquad \frac{1}{\mathrm{tr}\chi'(s)} \geq \frac{1}{\mathrm{tr}\chi'(0)} + \frac{s}{2}.$$

The right-hand side reaches 0 at $s = 2/|\mathrm{tr}\chi'(0)|$, hence $\mathrm{tr}\chi' \to -\infty$ before this affine time.

For a future trapped surface, the same argument applies to both future affine null normals. Penrose's global argument then implies future null geodesic incompleteness. □

2.1.1 Causality notions for Penrose's theorem

For $p \in \mathcal{M}$,

$$I^+(p) := \{q \in \mathcal{M} : \text{there exists a future-directed timelike curve from } p \text{ to } q\},$$

$$J^+(p) := \{q \in \mathcal{M} : \text{there exists a future-directed causal curve from } p \text{ to } q\}.$$

For $A \subset \mathcal{M}$,

$$I^+(A) := \bigcup_{p\in A} I^+(p), \qquad J^+(A) := \bigcup_{p\in A} J^+(p).$$

The past sets $I^-(p), J^-(p), I^-(A), J^-(A)$ are defined similarly.

A set $A \subset \mathcal{M}$ is achronal if

$$I^+(A) \cap A = \emptyset.$$

A hypersurface $\Sigma \subset \mathcal{M}$ is a Cauchy hypersurface if every inextendible timelike curve intersects Σ exactly once.

The spacetime $(\mathcal{M}, \mathbf{g})$ is globally hyperbolic if it admits a Cauchy hypersurface. A future-directed null geodesic is future complete if its affine parameter can be extended to $[0, +\infty)$. The spacetime is future null geodesically incomplete if there exists a future-directed null geodesic with finite affine length.

The future horismos of a set $A \subset \mathcal{M}$ is

$$E^+(A) := J^+(A) \setminus I^+(A) = \partial J^+(A).$$

2.1.2 Penrose's incompleteness theorem

Theorem 2.5 (Penrose incompleteness theorem)**.** *Let $(\mathcal{M}, \mathbf{g})$ be a globally hyperbolic spacetime. Assume:*

$$\mathbf{Ric}(V,V) \geq 0 \qquad \text{for every null vector } V,$$

$$\mathcal{M} \text{ contains a noncompact Cauchy hypersurface,}$$

and

$$\mathcal{M} \text{ contains a closed future trapped surface } S.$$

Then $(\mathcal{M}, \mathbf{g})$ is future null geodesically incomplete.

Remark 2.6. Raychaudhuri gives the local focusing estimate. If L' is an affine future null normal and $\mathrm{tr}\chi'(0) < 0$, then

$$L'(\mathrm{tr}\chi') = -\frac{1}{2}(\mathrm{tr}\chi')^2 - |\hat{\chi}'|^2 - \mathbf{Ric}(L', L') \leq -\frac{1}{2}(\mathrm{tr}\chi')^2,$$

so $\mathrm{tr}\chi' \to -\infty$ before affine time

$$\frac{2}{|\mathrm{tr}\chi'(0)|}.$$

For a future trapped surface, the same holds for both future null congruences.

The global part of Penrose's argument is causal: if all future null geodesics were complete, then $E^+(S) = \partial J^+(S)$ would be generated by future null geodesics orthogonal to S without conjugate points before leaving $E^+(S)$. The focusing estimate forces these generators to leave $E^+(S)$ in uniformly finite affine time, hence $E^+(S)$ is compact. This contradicts global hyperbolicity together with the existence of a noncompact Cauchy hypersurface. Therefore at least one future-directed null geodesic is incomplete.

2.2 Transversal equations

Proposition 2.7 (Mixed second variation equations)**.** *The transversal derivatives satisfy*

$$\begin{aligned} D(\Omega\underline{\chi})_{AB} = \Omega^2 &\left\{ \nabla\!\!\!/_A \eta_B + \nabla\!\!\!/_B \eta_A + 2\eta_A\eta_B + \frac{1}{2}(\chi \times \underline{\chi} + \underline{\chi} \times \chi)_{AB} + \rho \mathrm{g}\!\!\!/_{AB} \right\} \\ &+ \Omega^2 \left\{ \mathbf{Ric}_{AB} - \frac{1}{2}\mathbf{Ric}_{34} \mathrm{g}\!\!\!/_{AB} - \frac{\mathbf{R}}{3}\mathrm{g}\!\!\!/_{AB} \right\}, \end{aligned}$$

and

$$\begin{aligned} \underline{D}(\Omega\chi)_{AB} = \Omega^2 &\left\{ \nabla\!\!\!/_A \underline{\eta}_B + \nabla\!\!\!/_B \underline{\eta}_A + 2\underline{\eta}_A\underline{\eta}_B + \frac{1}{2}(\chi \times \underline{\chi} + \underline{\chi} \times \chi)_{AB} + \rho \mathrm{g}\!\!\!/_{AB} \right\} \\ &+ \Omega^2 \left\{ \mathbf{Ric}_{AB} - \frac{1}{2}\mathbf{Ric}_{34} \mathrm{g}\!\!\!/_{AB} - \frac{\mathbf{R}}{3}\mathrm{g}\!\!\!/_{AB} \right\}. \end{aligned}$$

Taking traces gives

$$D(\Omega \mathrm{tr}\underline{\chi}) = \Omega^2 \left\{ 2\mathrm{d}\!\!\!/\mathrm{iv}\eta + 2|\eta|^2 + (\chi, \underline{\chi}) + 2\rho + \mathrm{tr}_S \mathbf{Ric} - \mathbf{Ric}_{34} - \frac{2}{3}\mathbf{R} \right\},$$

$$\underline{D}(\Omega \mathrm{tr}\chi) = \Omega^2 \left\{ 2\mathrm{d}\!\!\!/\mathrm{iv}\underline{\eta} + 2|\underline{\eta}|^2 + (\chi, \underline{\chi}) + 2\rho + \mathrm{tr}_S \mathbf{Ric} - \mathbf{Ric}_{34} - \frac{2}{3}\mathbf{R} \right\}.$$

Proof. Commuting the e_4- and S-derivatives and using the frame formulas gives

$$\begin{aligned} D(\Omega\underline{\chi})_{AB} = \Omega^2 &\left\{ \nabla\!\!\!/_A \eta_B + \nabla\!\!\!/_B \eta_A + 2\eta_A\eta_B + \frac{1}{2}(\chi \times \underline{\chi} + \underline{\chi} \times \chi)_{AB} \right\} \\ &- \frac{\Omega^2}{2}\left(\mathbf{Rm}_{A4B3} + \mathbf{Rm}_{B4A3}\right). \end{aligned}$$

The curvature component is

$$\mathbf{Rm}_{A4B3} = \mathbf{W}_{A4B3} + \frac{1}{2}\left(\not{g}_{AB}\mathbf{Ric}_{43} - 2\mathbf{Ric}_{AB}\right) + \frac{\mathbf{R}}{3}\not{g}_{AB}$$
$$= -\rho\not{g}_{AB} + \sigma\not{\epsilon}_{AB} + \frac{1}{2}\mathbf{Ric}_{34}\not{g}_{AB} - \mathbf{Ric}_{AB} + \frac{\mathbf{R}}{3}\not{g}_{AB}.$$

Thus

$$\frac{1}{2}(\mathbf{Rm}_{A4B3} + \mathbf{Rm}_{B4A3}) = -\rho\not{g}_{AB} - \mathbf{Ric}_{AB} + \frac{1}{2}\mathbf{Ric}_{34}\not{g}_{AB} + \frac{\mathbf{R}}{3}\not{g}_{AB}.$$

Substitution gives the $D(\Omega\underline{\chi})$-equation. The $\underline{D}(\Omega\chi)$-equation is obtained by the same computation with η replaced by $\underline{\eta}$. Contracting with $\not{g}^{AB}$ gives the trace equations. □

Remark 2.8. These are the mixed second variation equations. They describe how the second fundamental form in one null direction changes along the other null direction. The terms involving $\eta, \underline{\eta}$ come from the normal connection.

2.3 Transport equations for the torsion one-forms

Proposition 2.9 (Transport equations for $\zeta, \eta, \underline{\eta}$)**.** *With*

$$D = \mathcal{L}_L, \qquad \underline{D} = \mathcal{L}_{\underline{L}}, \qquad L = \Omega e_4, \qquad \underline{L} = \Omega e_3,$$

the normal connection one-form ζ satisfies

$$(D\zeta)_A = -(\not{d}D\log\Omega)_A + \Omega\chi_A{}^B\eta_B - \Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A},$$

and

$$(\underline{D}\zeta)_A = (\not{d}\,\underline{D}\log\Omega)_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B - \Omega\underline{\beta}_A + \frac{\Omega}{2}\mathbf{Ric}_{3A}.$$

Consequently,

$$(D\eta)_A - \Omega\chi_A{}^B\eta_B = -\Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A},$$

and

$$(\underline{D}\underline{\eta})_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B = \Omega\underline{\beta}_A - \frac{\Omega}{2}\mathbf{Ric}_{3A}.$$

Proof. We prove the D-equation. Let X be tangent to $S_{u,\underline{u}}$ and Lie-transported by L:

$$[L, X] = 0.$$

Since $D = \mathcal{L}_L$ on S-one-forms,

$$(D\zeta)(X) = L(\zeta(X)).$$

Using

$$\zeta(X) = \frac{1}{2}\mathbf{g}(\nabla_X e_4, e_3),$$

we compute

$$(D\zeta)(X) = \frac{\Omega}{2}e_4\left[\mathbf{g}(\nabla_X e_4, e_3)\right]$$
$$= \frac{\Omega}{2}\mathbf{g}(\nabla_4\nabla_X e_4, e_3) + \frac{\Omega}{2}\mathbf{g}(\nabla_X e_4, \nabla_4 e_3).$$

Because $[L, X] = 0$,

$$[e_4, X] = (X\log\Omega)e_4.$$

Thus

$$\nabla_4\nabla_X e_4 = \nabla_X\nabla_4 e_4 + \nabla_{[e_4,X]}e_4 + R(e_4, X)e_4.$$

With our connection-coefficient convention,

$$\nabla_4 e_4 = -2\omega e_4, \qquad D\log\Omega = L(\log\Omega) = -2\Omega\omega.$$

Hence

$$\begin{aligned}\frac{\Omega}{2}\mathbf{g}(\nabla_X\nabla_4 e_4, e_3) &= \frac{\Omega}{2}\mathbf{g}(\nabla_X(-2\omega e_4), e_3)\\ &= 2\Omega X\omega - 2\Omega\omega\zeta(X),\end{aligned}$$

and

$$\frac{\Omega}{2}\mathbf{g}(\nabla_{[e_4,X]}e_4, e_3) = \frac{\Omega}{2}(X\log\Omega)g(\nabla_4 e_4, e_3) = 2\Omega\omega X(\log\Omega).$$

The remaining connection term is

$$\begin{aligned}\frac{\Omega}{2}\mathbf{g}(\nabla_X e_4, \nabla_4 e_3) &= \frac{\Omega}{2}g(\chi(X)^{\#} - \zeta(X)e_4, 2\omega e_3 + 2\underline{\eta}^{\#})\\ &= \Omega\chi(X, \underline{\eta}^{\#}) + 2\Omega\omega\zeta(X).\end{aligned}$$

The $\omega\zeta$ terms cancel, and therefore

$$(D\zeta)(X) = 2\Omega X\omega + 2\Omega\omega X(\log\Omega) + \Omega\chi(X, \underline{\eta}^{\#}) + \frac{\Omega}{2}\mathbf{Rm}_{4X43}.$$

Since

$$D\log\Omega = -2\Omega\omega,$$

we have

$$-X(D\log\Omega) = 2\Omega X\omega + 2\Omega\omega X(\log\Omega).$$

Also,

$$\underline{\eta} = -\zeta + \not{d}\log\Omega, \qquad \eta = \zeta + \not{d}\log\Omega,$$

so

$$\underline{\eta} + 2\zeta = \eta.$$

Using the frame identity

$$\nabla_4 e_3 = 2\omega e_3 + 2\underline{\eta}^{\#},$$

the preceding computation is equivalently

$$(D\zeta)(X) = -(\not{d}D\log\Omega)(X) + \Omega\chi(X, \eta^{\#}) - \frac{\Omega}{2}\mathbf{Rm}_{X434}.$$

Thus

$$(D\zeta)_A = -(\not{d}D\log\Omega)_A + \Omega\chi_A{}^B\eta_B - \frac{\Omega}{2}\mathbf{Rm}_{A434}.$$

Using

$$\mathbf{Rm}_{A434} = 2\beta_A + \mathbf{Ric}_{4A},$$

we obtain

$$(D\zeta)_A = -(\not{d}D\log\Omega)_A + \Omega\chi_A{}^B\eta_B - \Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A}.$$

The $\underline{D}$-equation is the conjugate computation. It gives

$$(\underline{D}\zeta)_A = (\not{d}\,\underline{D}\log\Omega)_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B - \frac{\Omega}{2}\mathbf{Rm}_{A334}.$$

Since

$$\mathbf{Rm}_{A334} = 2\underline{\beta}_A - \mathbf{Ric}_{3A},$$

we get

$$(\underline{D}\zeta)_A = (\nabla\!\!\!/\, \underline{D}\log\Omega)_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B - \Omega\underline{\beta}_A + \frac{\Omega}{2}\mathbf{Ric}_{3A}.$$

Finally, from

$$\eta = \zeta + \nabla\!\!\!/ \log\Omega, \qquad \underline{\eta} = -\zeta + \nabla\!\!\!/ \log\Omega,$$

and

$$D\nabla\!\!\!/ f = \nabla\!\!\!/ Df, \qquad \underline{D}\nabla\!\!\!/ f = \nabla\!\!\!/\, \underline{D}f,$$

we obtain

$$D\eta = D\zeta + \nabla\!\!\!/ D\log\Omega,$$

hence

$$(D\eta)_A = \Omega\chi_A{}^B\eta_B - \Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A}.$$

Therefore

$$(D\eta)_A - \Omega\chi_A{}^B\eta_B = -\Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A}.$$

Similarly,

$$\underline{D}\underline{\eta} = -\underline{D}\zeta + \nabla\!\!\!/\, \underline{D}\log\Omega,$$

so

$$(\underline{D}\underline{\eta})_A = \Omega\underline{\chi}_A{}^B\underline{\eta}_B + \Omega\underline{\beta}_A - \frac{\Omega}{2}\mathbf{Ric}_{3A},$$

and hence

$$(\underline{D}\underline{\eta})_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B = \Omega\underline{\beta}_A - \frac{\Omega}{2}\mathbf{Ric}_{3A}.$$

□

2.4 Gauss–Codazzi–Ricci equations

Proposition 2.10 (Gauss–Codazzi–Ricci equations)**.** *The Gauss equation is*

$$K = -\rho - \frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) + \frac{1}{2}(\hat{\chi}, \hat{\underline{\chi}}) + \frac{1}{2}\mathrm{tr}_S\mathbf{Ric} - \frac{1}{6}\mathbf{R}.$$

The Codazzi equations are

$$(\mathrm{d}\!\!\!/\mathrm{iv}\chi)_A - (\nabla\!\!\!/ \mathrm{tr}\chi)_A + \chi_A{}^B\zeta_B - (\mathrm{tr}\chi)\zeta_A = -\beta_A + \frac{1}{2}\mathbf{Ric}_{4A},$$

and

$$(\mathrm{d}\!\!\!/\mathrm{iv}\underline{\chi})_A - (\nabla\!\!\!/ \mathrm{tr}\underline{\chi})_A - \underline{\chi}_A{}^B\zeta_B + (\mathrm{tr}\underline{\chi})\zeta_A = \underline{\beta}_A + \frac{1}{2}\mathbf{Ric}_{3A}.$$

The Ricci equation for the normal bundle is

$$\mathrm{c\!\!\!/url}\zeta = \frac{1}{2}\chi \wedge \underline{\chi} - \sigma.$$

Consequently,

$$\mathrm{c\!\!\!/url}\eta = \frac{1}{2}\chi \wedge \underline{\chi} - \sigma, \qquad \mathrm{c\!\!\!/url}\underline{\eta} = -\frac{1}{2}\chi \wedge \underline{\chi} + \sigma.$$

Proof. For $X, Y \in TS_{u,\underline{u}}$, write

$$\nabla_X Y = \nabla\!\!\!/_X Y + \mathrm{I\!I}(X,Y), \qquad \mathrm{I\!I}(X,Y) \in NS_{u,\underline{u}}.$$

For $N \in \Gamma(NS_{u,\underline{u}})$, define

$$\nabla^\perp_X N := (\nabla_X N)^\perp, \qquad \mathbf{g}(A_N X, Y) = \mathbf{g}(\mathrm{I\!I}(X,Y), N).$$

With Christodoulou's curvature convention

$$\mathbf{Rm}(W, Z, X, Y) = \mathbf{g}(W, R(X, Y)Z),$$

the standard Gauss–Codazzi–Ricci equations are

$$\begin{aligned}\mathbf{Rm}^S(W, Z, X, Y) = \mathbf{Rm}(W, Z, X, Y) \\ + \mathbf{g}(\mathbb{II}(W, X), \mathbb{II}(Z, Y)) - \mathbf{g}(\mathbb{II}(W, Y), \mathbb{II}(Z, X)),\end{aligned}$$

$$(\nabla^\perp_X \mathbb{II})(Y, Z) - (\nabla^\perp_Y \mathbb{II})(X, Z) = (R(X, Y)Z)^\perp,$$

and

$$\mathbf{g}(R^\perp(X, Y)N, M) = \mathbf{Rm}(N, M, X, Y) + \mathbf{g}([A_N, A_M]X, Y).$$

Since

$$\chi(X, Y) = \mathbf{g}(\nabla_X e_4, Y), \qquad \underline{\chi}(X, Y) = \mathbf{g}(\nabla_X e_3, Y),$$

we have

$$\mathbf{g}(\mathbb{II}(X, Y), e_4) = -\chi(X, Y), \qquad \mathbf{g}(\mathbb{II}(X, Y), e_3) = -\underline{\chi}(X, Y).$$

Since $\mathbf{g}(e_3, e_4) = -2$,

$$\mathbb{II}(X, Y) = \frac{1}{2}\chi(X, Y)e_3 + \frac{1}{2}\underline{\chi}(X, Y)e_4.$$

Moreover,

$$\nabla^\perp_A e_4 = -\zeta_A e_4, \qquad \nabla^\perp_A e_3 = \zeta_A e_3.$$

Gauss equation. For

$$\mathbb{II}_{AB} = \frac{1}{2}\chi_{AB} e_3 + \frac{1}{2}\underline{\chi}_{AB} e_4,$$

we have

$$\mathbf{g}(\mathbb{II}_{AC}, \mathbb{II}_{BD}) = -\frac{1}{2}\big(\chi_{AC}\underline{\chi}_{BD} + \underline{\chi}_{AC}\chi_{BD}\big),$$

and

$$\mathbf{g}(\mathbb{II}_{AD}, \mathbb{II}_{BC}) = -\frac{1}{2}\big(\chi_{AD}\underline{\chi}_{BC} + \underline{\chi}_{AD}\chi_{BC}\big).$$

Therefore the standard Gauss equation gives

$$\begin{aligned}\mathbf{Rm}^S_{ABCD} = \mathbf{Rm}_{ABCD} \\ - \frac{1}{2}\big(\chi_{AC}\underline{\chi}_{BD} + \underline{\chi}_{AC}\chi_{BD} - \chi_{AD}\underline{\chi}_{BC} - \underline{\chi}_{AD}\chi_{BC}\big).\end{aligned}$$

Equivalently,

$$\begin{aligned}\mathbf{Rm}^S_{ABCD} + \frac{1}{2}\big(\chi_{AC}\underline{\chi}_{BD} + \underline{\chi}_{AC}\chi_{BD} - \chi_{AD}\underline{\chi}_{BC} - \underline{\chi}_{AD}\chi_{BC}\big) \\ = \mathbf{Rm}_{ABCD}.\end{aligned}$$

Since

$$\mathbf{Rm}^S_{ABCD} = K(\not{g}_{AC}\not{g}_{BD} - \not{g}_{AD}\not{g}_{BC}),$$

we have

$$\frac{1}{4}\not{\epsilon}^{AB}\not{\epsilon}^{CD}\mathbf{Rm}^S_{ABCD} = K.$$

Using

$$\not{\epsilon}^{AB}\not{\epsilon}^{CD} = \not{g}^{AC}\not{g}^{BD} - \not{g}^{AD}\not{g}^{BC},$$

we compute

$$\begin{aligned}
&\frac{1}{8}\not{\epsilon}^{AB}\not{\epsilon}^{CD}\big(\chi_{AC}\underline{\chi}_{BD}+\underline{\chi}_{AC}\chi_{BD}-\chi_{AD}\underline{\chi}_{BC}-\underline{\chi}_{AD}\chi_{BC}\big)\\
&=\frac{1}{8}\big(\not{g}^{AC}\not{g}^{BD}-\not{g}^{AD}\not{g}^{BC}\big)\big(\chi_{AC}\underline{\chi}_{BD}+\underline{\chi}_{AC}\chi_{BD}-\chi_{AD}\underline{\chi}_{BC}-\underline{\chi}_{AD}\chi_{BC}\big)\\
&=\frac{1}{8}\big(2(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi})-2(\chi,\underline{\chi})-2(\chi,\underline{\chi})+2(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi})\big)\\
&=\frac{1}{2}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi})-\frac{1}{2}(\chi,\underline{\chi}).
\end{aligned}$$

Thus

$$K+\frac{1}{2}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi})-\frac{1}{2}(\chi,\underline{\chi})=\frac{1}{4}\not{\epsilon}^{AB}\not{\epsilon}^{CD}\mathbf{Rm}_{ABCD}.$$

The curvature decomposition gives

$$\frac{1}{4}\not{\epsilon}^{AB}\not{\epsilon}^{CD}\mathbf{Rm}_{ABCD}=-\rho+\frac{1}{2}\mathrm{tr}_S\mathbf{Ric}-\frac{1}{6}\mathbf{R}.$$

Therefore

$$K=-\rho-\frac{1}{2}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi})+\frac{1}{2}(\chi,\underline{\chi})+\frac{1}{2}\mathrm{tr}_S\mathbf{Ric}-\frac{1}{6}\mathbf{R}.$$

Using

$$(\chi,\underline{\chi})=(\hat{\chi},\hat{\underline{\chi}})+\frac{1}{2}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}),$$

we obtain

$$K=-\rho-\frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi})+\frac{1}{2}(\hat{\chi},\hat{\underline{\chi}})+\frac{1}{2}\mathrm{tr}_S\mathbf{Ric}-\frac{1}{6}\mathbf{R}.$$

Outgoing Codazzi equation. For $X=e_A, Y=e_B, Z=e_C$, pair the standard Codazzi equation with e_4. Since

$$\mathbf{g}(\mathbb{II}_{BC},e_4)=-\chi_{BC},\qquad \nabla^{\perp}_A e_4=-\zeta_A e_4,$$

we have

$$\begin{aligned}
\mathbf{g}((\nabla^{\perp}_A\mathbb{II})(e_B,e_C),e_4)&=\not{\nabla}_A\mathbf{g}(\mathbb{II}_{BC},e_4)-\mathbf{g}(\mathbb{II}_{BC},\nabla^{\perp}_A e_4)\\
&=-\not{\nabla}_A\chi_{BC}-\zeta_A\chi_{BC}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{g}((\nabla^{\perp}_A\mathbb{II})(e_B,e_C),e_4)&=\mathbf{g}(\nabla^{\perp}_A(\mathbb{II}_{BC}),e_4)-\mathbf{g}(\mathbb{II}(\not{\nabla}_A e_B,e_C),e_4)-\mathbf{g}(\mathbb{II}(e_B,\not{\nabla}_A e_C),e_4)\\
&=e_A\mathbf{g}(\mathbb{II}_{BC},e_4)-\mathbf{g}(\mathbb{II}_{BC},\nabla^{\perp}_A e_4)\\
&\quad-\mathbf{g}(\mathbb{II}(\not{\nabla}_A e_B,e_C),e_4)-\mathbf{g}(\mathbb{II}(e_B,\not{\nabla}_A e_C),e_4)\\
&=\not{\nabla}_A\big(\mathbf{g}(\mathbb{II}_{BC},e_4)\big)-\mathbf{g}(\mathbb{II}_{BC},\nabla^{\perp}_A e_4)\\
&=-\not{\nabla}_A\chi_{BC}-\mathbf{g}(\mathbb{II}_{BC},-\zeta_A e_4)\\
&=-\not{\nabla}_A\chi_{BC}+\zeta_A\mathbf{g}(\mathbb{II}_{BC},e_4)\\
&=-\not{\nabla}_A\chi_{BC}-\zeta_A\chi_{BC}.
\end{aligned}$$

Thus

$$\begin{aligned}
&-\not{\nabla}_A\chi_{BC}+\not{\nabla}_B\chi_{AC}-\zeta_A\chi_{BC}+\zeta_B\chi_{AC}\\
&\quad=\mathbf{g}(R(e_A,e_B)e_C,e_4).
\end{aligned}$$

By Christodoulou's convention,

$$\mathbf{g}(R(e_A,e_B)e_C,e_4)=\mathbf{Rm}_{4CAB}=-\mathbf{Rm}_{C4AB}.$$

Hence

$$\not{\nabla}_A\chi_{BC}-\not{\nabla}_B\chi_{AC}=\mathbf{Rm}_{C4AB}-\zeta_A\chi_{BC}+\zeta_B\chi_{AC}.$$

Contracting A, C, we get

$$(\mathrm{d}\!\!/\mathrm{iv}\chi)_B - (\nabla\!\!\!/\mathrm{tr}\chi)_B = g\!\!\!/^{AC}\mathbf{Rm}_{C4AB} - \zeta^A\chi_{BC} + \zeta_B\mathrm{tr}\chi.$$

Therefore

$$(\mathrm{d}\!\!/\mathrm{iv}\chi)_B - (\nabla\!\!\!/\mathrm{tr}\chi)_B + \chi_B{}^A\zeta_A - (\mathrm{tr}\chi)\zeta_B = g\!\!\!/^{AC}\mathbf{Rm}_{C4AB}.$$

Now

$$\begin{aligned} g\!\!\!/^{AC}\mathbf{Rm}_{C4AB} &= g\!\!\!/^{AC}\mathbf{W}_{C4AB} + \frac{1}{2}g\!\!\!/^{AC}\big(g\!\!\!/_{CA}\mathbf{Ric}_{4B} - g\!\!\!/_{CB}\mathbf{Ric}_{4A}\big) \\ &= g\!\!\!/^{AC}\mathbf{W}_{C4AB} + \frac{1}{2}\mathbf{Ric}_{4B}. \end{aligned}$$

In vacuum Christodoulou's Codazzi equation gives

$$g\!\!\!/^{AC}\mathbf{W}_{C4AB} = -\beta_B,$$

and hence in general

$$g\!\!\!/^{AC}\mathbf{Rm}_{C4AB} = -\beta_B + \frac{1}{2}\mathbf{Ric}_{4B}.$$

Thus

$$(\mathrm{d}\!\!/\mathrm{iv}\chi)_B - (\nabla\!\!\!/\mathrm{tr}\chi)_B + \chi_B{}^A\zeta_A - (\mathrm{tr}\chi)\zeta_B = -\beta_B + \frac{1}{2}\mathbf{Ric}_{4B}.$$

By conjugation,

$$(\mathrm{d}\!\!/\mathrm{iv}\underline{\chi})_A - (\nabla\!\!\!/\mathrm{tr}\underline{\chi})_A - \underline{\chi}_A{}^B\zeta_B + (\mathrm{tr}\underline{\chi})\zeta_A = \underline{\beta}_A + \frac{1}{2}\mathbf{Ric}_{3A}.$$

Ricci equation for the normal bundle. For $X, Y \in TS_{u,\underline{u}}$ and $N, M \in NS_{u,\underline{u}}$, the standard Ricci equation is

$$\mathbf{g}(R^\perp(X,Y)N, M) = \mathbf{Rm}(M,N,X,Y) + \mathbf{g}([A_N, A_M]X, Y),$$

where

$$\mathbf{g}(A_N X, Y) = \mathbf{g}(\mathbb{II}(X,Y), N).$$

Taking

$$X = e_A, \qquad Y = e_B, \qquad N = e_4, \qquad M = e_3,$$

we get

$$\mathbf{g}(R^\perp_{AB}e_4, e_3) = \mathbf{Rm}_{34AB} + \mathbf{g}([A_{e_4}, A_{e_3}]e_A, e_B).$$

By the pair symmetry of $\mathbf{Rm}$,

$$\mathbf{Rm}_{34AB} = \mathbf{Rm}_{AB34}.$$

We first compute the left-hand side. Using

$$\nabla^\perp_A e_4 = -\zeta_A e_4,$$

we have

$$\begin{aligned} R^\perp_{AB}e_4 &= \nabla^\perp_A\nabla^\perp_B e_4 - \nabla^\perp_B\nabla^\perp_A e_4 - \nabla^\perp_{[e_A,e_B]}e_4 \\ &= \nabla^\perp_A(-\zeta_B e_4) - \nabla^\perp_B(-\zeta_A e_4) - \nabla^\perp_{[e_A,e_B]}e_4 \\ &= -e_A(\zeta_B)e_4 - \zeta_B\nabla^\perp_A e_4 + e_B(\zeta_A)e_4 + \zeta_A\nabla^\perp_B e_4 - \nabla^\perp_{[e_A,e_B]}e_4 \\ &= -e_A(\zeta_B)e_4 + \zeta_A\zeta_B e_4 + e_B(\zeta_A)e_4 - \zeta_A\zeta_B e_4 + \zeta([e_A,e_B])e_4 \\ &= -\big(e_A(\zeta_B) - e_B(\zeta_A) - \zeta([e_A,e_B])\big)e_4 \\ &= -(\nabla\!\!\!/_A\zeta_B - \nabla\!\!\!/_B\zeta_A)e_4. \end{aligned}$$

Therefore

$$\mathbf{g}(R^\perp_{AB}e_4, e_3) = 2(\nabla\!\!\!/_A\zeta_B - \nabla\!\!\!/_B\zeta_A).$$

Next,

$$A_{e_4} = -\chi^\sharp, \qquad A_{e_3} = -\underline{\chi}^\sharp.$$

Indeed,

$$\mathbf{g}(A_{e_4}X,Y) = \mathbf{g}(\mathrm{I\!I}(X,Y),e_4) = -\chi(X,Y),$$

and similarly

$$\mathbf{g}(A_{e_3}X,Y) = -\underline{\chi}(X,Y).$$

Thus

$$\begin{aligned}
\mathbf{g}([A_{e_4},A_{e_3}]e_A,e_B) &= \mathbf{g}(A_{e_4}A_{e_3}e_A,e_B) - \mathbf{g}(A_{e_3}A_{e_4}e_A,e_B) \\
&= \mathbf{g}(A_{e_4}(-\underline{\chi}_A{}^C e_C),e_B) - \mathbf{g}(A_{e_3}(-\chi_A{}^C e_C),e_B) \\
&= -\underline{\chi}_A{}^C\mathbf{g}(A_{e_4}e_C,e_B) + \chi_A{}^C\mathbf{g}(A_{e_3}e_C,e_B) \\
&= -\underline{\chi}_A{}^C(-\chi_{CB}) + \chi_A{}^C(-\underline{\chi}_{CB}) \\
&= \underline{\chi}_A{}^C\chi_{CB} - \chi_A{}^C\underline{\chi}_{CB}.
\end{aligned}$$

Hence the Ricci equation gives

$$2(\nabla\!\!\!/_A\zeta_B - \nabla\!\!\!/_B\zeta_A) = \mathbf{Rm}_{AB34} + \underline{\chi}_A{}^C\chi_{CB} - \chi_A{}^C\underline{\chi}_{CB}.$$

Contracting with $\frac{1}{2}\epsilon\!\!/^{AB}$,

$$\begin{aligned}
\mathrm{cu\!\!/rl}\zeta &= \frac{1}{2}\epsilon\!\!/^{AB}(\nabla\!\!\!/_A\zeta_B - \nabla\!\!\!/_B\zeta_A) \\
&= \frac{1}{4}\epsilon\!\!/^{AB}\mathbf{Rm}_{AB34} + \frac{1}{4}\epsilon\!\!/^{AB}\big(\underline{\chi}_A{}^C\chi_{CB} - \chi_A{}^C\underline{\chi}_{CB}\big).
\end{aligned}$$

For the second fundamental form term,

$$\epsilon\!\!/^{AB}\underline{\chi}_A{}^C\chi_{CB} = -\epsilon\!\!/^{AB}\chi_A{}^C\underline{\chi}_{CB}.$$

Therefore

$$\frac{1}{4}\epsilon\!\!/^{AB}\big(\underline{\chi}_A{}^C\chi_{CB} - \chi_A{}^C\underline{\chi}_{CB}\big) = -\frac{1}{2}\epsilon\!\!/^{AB}\chi_A{}^C\underline{\chi}_{CB} = -\frac{1}{2}\chi\wedge\underline{\chi}.$$

The Ricci and scalar terms in $\mathbf{Rm}_{AB34}$ vanish after contraction by $\epsilon\!\!/^{AB}$, hence

$$\frac{1}{4}\epsilon\!\!/^{AB}\mathbf{Rm}_{AB34} = \frac{1}{4}\epsilon\!\!/^{AB}\mathbf{W}_{AB34} = \sigma.$$

Thus

$$\mathrm{cu\!\!/rl}\zeta = \sigma - \frac{1}{2}\chi\wedge\underline{\chi}.$$

Finally, since

$$\eta = \zeta + d\!\!/\log\Omega, \qquad \underline{\eta} = -\zeta + d\!\!/\log\Omega, \qquad \mathrm{cu\!\!/rl}\,d\!\!/ f = 0,$$

we get

$$\mathrm{cu\!\!/rl}\eta = \sigma - \frac{1}{2}\chi\wedge\underline{\chi}, \qquad \mathrm{cu\!\!/rl}\underline{\eta} = -\sigma + \frac{1}{2}\chi\wedge\underline{\chi}.$$

□

Remark 2.11. In vacuum, these become

$$K = -\rho - \frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) + \frac{1}{2}(\hat{\chi},\hat{\underline{\chi}}),$$

$$\mathrm{di\!\!/v}\chi - d\!\!/\mathrm{tr}\chi + \chi\cdot\zeta - (\mathrm{tr}\chi)\zeta = -\beta, \qquad \mathrm{di\!\!/v}\underline{\chi} - d\!\!/\mathrm{tr}\underline{\chi} - \underline{\chi}\cdot\zeta + (\mathrm{tr}\underline{\chi})\zeta = \underline{\beta},$$

and

$$\mathrm{cu\!\!/rl}\zeta = \frac{1}{2}\chi\wedge\underline{\chi} - \sigma.$$

2.5 The collected null structure system

The full system consists of the first variation equations

$$D\not{g} = 2\Omega\chi, \qquad \underline{D}\not{g} = 2\Omega\underline{\chi},$$

$$D(d\mu_{\not{g}}) = \Omega(\mathrm{tr}\chi)d\mu_{\not{g}}, \qquad \underline{D}(d\mu_{\not{g}}) = \Omega(\mathrm{tr}\underline{\chi})d\mu_{\not{g}},$$

and the structure equations

$$D\chi = -2\Omega\omega\,\chi + \Omega\chi\times\chi - \Omega\alpha - \frac{\Omega}{2}\mathbf{Ric}_{44}\not{g},$$

$$\underline{D}\underline{\chi} = -2\Omega\underline{\omega}\,\underline{\chi} + \Omega\underline{\chi}\times\underline{\chi} - \Omega\underline{\alpha} - \frac{\Omega}{2}\mathbf{Ric}_{33}\not{g},$$

$$D\mathrm{tr}\chi = -\frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \Omega|\hat{\chi}|^2 - 2\Omega\omega\,\mathrm{tr}\chi - \Omega\mathbf{Ric}_{44},$$

$$\underline{D}\mathrm{tr}\underline{\chi} = -\frac{\Omega}{2}(\mathrm{tr}\underline{\chi})^2 - \Omega|\hat{\underline{\chi}}|^2 - 2\Omega\underline{\omega}\,\mathrm{tr}\underline{\chi} - \Omega\mathbf{Ric}_{33},$$

$$\begin{aligned} D(\Omega\underline{\chi})_{AB} = \Omega^2 &\left\{\not{\nabla}_A\eta_B + \not{\nabla}_B\eta_A + 2\eta_A\eta_B + \frac{1}{2}(\chi\times\underline{\chi} + \underline{\chi}\times\chi)_{AB} + \rho\not{g}_{AB}\right\} \\ &+ \Omega^2\left\{\mathbf{Ric}_{AB} - \frac{1}{2}\mathbf{Ric}_{34}\not{g}_{AB} - \frac{\mathbf{R}}{3}\not{g}_{AB}\right\}, \end{aligned}$$

$$\begin{aligned} \underline{D}(\Omega\chi)_{AB} = \Omega^2 &\left\{\not{\nabla}_A\underline{\eta}_B + \not{\nabla}_B\underline{\eta}_A + 2\underline{\eta}_A\underline{\eta}_B + \frac{1}{2}(\chi\times\underline{\chi} + \underline{\chi}\times\chi)_{AB} + \rho\not{g}_{AB}\right\} \\ &+ \Omega^2\left\{\mathbf{Ric}_{AB} - \frac{1}{2}\mathbf{Ric}_{34}\not{g}_{AB} - \frac{\mathbf{R}}{3}\not{g}_{AB}\right\}, \end{aligned}$$

$$D(\Omega\mathrm{tr}\underline{\chi}) = \Omega^2\left\{2\not{\mathrm{d}}\mathrm{iv}\eta + 2|\eta|^2 + (\chi,\underline{\chi}) + 2\rho + \mathrm{tr}_S\mathbf{Ric} - \mathbf{Ric}_{34} - \frac{2}{3}\mathbf{R}\right\},$$

$$\underline{D}(\Omega\mathrm{tr}\chi) = \Omega^2\left\{2\not{\mathrm{d}}\mathrm{iv}\underline{\eta} + 2|\underline{\eta}|^2 + (\chi,\underline{\chi}) + 2\rho + \mathrm{tr}_S\mathbf{Ric} - \mathbf{Ric}_{34} - \frac{2}{3}\mathbf{R}\right\},$$

$$(D\eta)_A - \Omega\chi_A{}^B\eta_B = -\Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A},$$

$$(\underline{D}\underline{\eta})_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B = \Omega\underline{\beta}_A - \frac{\Omega}{2}\mathbf{Ric}_{3A}.$$

$$\not{\mathrm{c}}\mathrm{url}\eta = \frac{1}{2}\chi\wedge\underline{\chi} - \sigma, \qquad \not{\mathrm{c}}\mathrm{url}\underline{\eta} = -\frac{1}{2}\chi\wedge\underline{\chi} + \sigma, \qquad \not{\mathrm{c}}\mathrm{url}\zeta = \frac{1}{2}\chi\wedge\underline{\chi} - \sigma,$$

$$K = -\rho - \frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) + \frac{1}{2}(\hat{\chi},\hat{\underline{\chi}}) + \frac{1}{2}\mathrm{tr}_S\mathbf{Ric} - \frac{1}{6}\mathbf{R},$$

$$(\not{\mathrm{d}}\mathrm{iv}\chi)_A - (\not{d}\mathrm{tr}\chi)_A + \chi_A{}^B\zeta_B - (\mathrm{tr}\chi)\zeta_A = -\beta_A - \frac{1}{2}\mathbf{Ric}_{4A},$$

$$(\not{\mathrm{d}}\mathrm{iv}\underline{\chi})_A - (\not{d}\mathrm{tr}\underline{\chi})_A - \underline{\chi}_A{}^B\zeta_B + (\mathrm{tr}\underline{\chi})\zeta_A = \underline{\beta}_A - \frac{1}{2}\mathbf{Ric}_{3A}.$$

2.6 Derived transport equations

The following equations are consequences of the basic torsion transport equations, the lapse identities

$$D \log \Omega = -2\Omega\omega, \qquad \underline{D} \log \Omega = -2\Omega\underline{\omega},$$

and the Ricci identities for the null frame. We set

$$\rho_{\mathbf{Rm}} = \frac{1}{4}\mathbf{Rm}_{3434} = \rho + \frac{1}{2}\mathbf{Ric}_{34} + \frac{1}{6}\mathbf{R}.$$

Corollary 2.12 (Derived torsion transports)**.** *The torsion one-forms satisfy*

$$(D\eta)_A - \Omega\chi_A{}^B\eta_B = -\Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A},$$

$$(\underline{D}\underline{\eta})_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B = \Omega\underline{\beta}_A - \frac{\Omega}{2}\mathbf{Ric}_{3A},$$

and

$$(D\underline{\eta})_A = -4(\not{d}(\Omega\omega))_A - \Omega\chi_A{}^B\eta_B + \Omega\beta_A + \frac{\Omega}{2}\mathbf{Ric}_{4A},$$

$$(\underline{D}\eta)_A = -4(\not{d}(\Omega\underline{\omega}))_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B - \Omega\underline{\beta}_A + \frac{\Omega}{2}\mathbf{Ric}_{3A}.$$

Proof. The transport equations for ζ are

$$(D\zeta)_A = 2(\not{d}(\Omega\omega))_A + \Omega\chi_A{}^B\eta_B - \Omega\beta_A - \frac{\Omega}{2}\mathbf{Ric}_{4A},$$

$$(\underline{D}\zeta)_A = -2(\not{d}(\Omega\underline{\omega}))_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B - \Omega\underline{\beta}_A + \frac{\Omega}{2}\mathbf{Ric}_{3A}.$$

Since

$$\eta = \zeta + \not{d}\log\Omega, \qquad \underline{\eta} = -\zeta + \not{d}\log\Omega,$$

and

$$D\not{d}f = \not{d}Df, \qquad \underline{D}\not{d}f = \not{d}\,\underline{D}f,$$

we have

$$D\eta = D\zeta + \not{d}D\log\Omega = D\zeta - 2\not{d}(\Omega\omega),$$
$$D\underline{\eta} = -D\zeta + \not{d}D\log\Omega = -D\zeta - 2\not{d}(\Omega\omega),$$

and similarly

$$\underline{D}\underline{\eta} = -\underline{D}\zeta + \not{d}\,\underline{D}\log\Omega = -\underline{D}\zeta - 2\not{d}(\Omega\underline{\omega}),$$
$$\underline{D}\eta = \underline{D}\zeta + \not{d}\,\underline{D}\log\Omega = \underline{D}\zeta - 2\not{d}(\Omega\underline{\omega}).$$

Substituting the two equations for $D\zeta$ and $\underline{D}\zeta$ gives the four displayed identities. □

Corollary 2.13 (Derived lapse equations)**.** *Set*

$$\rho_{\mathbf{Rm}} = \frac{1}{4}\mathbf{Rm}_{3434} = \rho + \frac{1}{2}\mathbf{Ric}_{34} + \frac{1}{6}\mathbf{R}.$$

Then the mixed lapse transports are

$$D\underline{\omega} = \Omega\left(2\omega\underline{\omega} - (\eta, \underline{\eta}) + \frac{1}{2}|\eta|^2 - \frac{1}{2}\rho_{\mathbf{Rm}}\right),$$

and

$$\underline{D}\omega = \Omega\left(2\omega\underline{\omega} - (\eta, \underline{\eta}) + \frac{1}{2}|\underline{\eta}|^2 - \frac{1}{2}\rho_{\mathbf{Rm}}\right).$$

The same-direction derivatives are only the identities

$$D\omega = 2\Omega\omega^2 - \frac{1}{2\Omega}D^2 \log\Omega, \qquad \underline{D}\underline{\omega} = 2\Omega\underline{\omega}^2 - \frac{1}{2\Omega}\underline{D}^2 \log\Omega.$$

Thus $D\omega$ *and* $\underline{D}\underline{\omega}$ *are not curvature-controlled structure equations unless one imposes an additional equation for the second derivatives of the lapse.*

Proof. We first derive the sum of $D\underline{\omega}$ and $\underline{D}\omega$. By definition,

$$\underline{\omega} = -\frac{1}{4}\mathbf{g}(\nabla_3 e_4, e_3),$$

so

$$\begin{aligned} D\underline{\omega} &= -\frac{\Omega}{4} e_4 \left[g(\nabla_3 e_4, e_3)\right] \\ &= -\frac{\Omega}{4}\left\{g(\nabla_4\nabla_3 e_4, e_3) + g(\nabla_3 e_4, \nabla_4 e_3)\right\}. \end{aligned}$$

Using the curvature convention

$$R(e_4, e_3)e_4 = \nabla_4\nabla_3 e_4 - \nabla_3\nabla_4 e_4 - \nabla_{[e_4,e_3]} e_4,$$

we write

$$\nabla_4\nabla_3 e_4 = \nabla_3\nabla_4 e_4 + \nabla_{[e_4,e_3]} e_4 + R(e_4, e_3)e_4.$$

The frame formulas are

$$\nabla_4 e_4 = -2\omega e_4, \qquad \nabla_3 e_4 = 2\underline{\omega} e_4 + 2\eta^{\#},$$

$$\nabla_4 e_3 = 2\omega e_3 + 2\underline{\eta}^{\#}, \qquad [e_4, e_3] = 2(\underline{\eta} - \eta)^{\#} + 2\omega e_3 - 2\underline{\omega} e_4.$$

We compute the three pieces. First,

$$\begin{aligned} g(\nabla_3\nabla_4 e_4, e_3) &= g(\nabla_3(-2\omega e_4), e_3) \\ &= -2e_3(\omega) g(e_4, e_3) - 2\omega g(\nabla_3 e_4, e_3) \\ &= 4e_3(\omega) + 8\omega\underline{\omega}. \end{aligned}$$

Second,

$$\begin{aligned} g(\nabla_{[e_4,e_3]} e_4, e_3) &= g(\nabla_{2(\underline{\eta}-\eta)^{\#}+2\omega e_3 - 2\underline{\omega} e_4} e_4, e_3) \\ &= 4\zeta(\underline{\eta}^{\#} - \eta^{\#}) - 8\omega\underline{\omega} - 8\omega\underline{\omega} \\ &= -2|\eta - \underline{\eta}|^2 - 16\omega\underline{\omega}. \end{aligned}$$

Here we used

$$\zeta = \frac{1}{2}(\eta - \underline{\eta}).$$

Third,

$$g(R(e_4, e_3)e_4, e_3) = \mathbf{Rm}_{4343} = \mathbf{Rm}_{3434} = 4\rho_{\mathbf{Rm}}.$$

Finally,

$$\begin{aligned} g(\nabla_3 e_4, \nabla_4 e_3) &= g(2\underline{\omega} e_4 + 2\eta^{\#}, 2\omega e_3 + 2\underline{\eta}^{\#}) \\ &= -8\omega\underline{\omega} + 4(\eta, \underline{\eta}). \end{aligned}$$

Substituting these four identities gives

$$\begin{aligned} D\underline{\omega} &= -\frac{\Omega}{4}\Big[4e_3(\omega) - 16\omega\underline{\omega} - 2|\eta - \underline{\eta}|^2 + 4\rho_{\mathbf{Rm}} + 4(\eta, \underline{\eta})\Big] \\ &= -\underline{D}\omega + 4\Omega\omega\underline{\omega} + \frac{\Omega}{2}|\eta - \underline{\eta}|^2 - \Omega\rho_{\mathbf{Rm}} - \Omega(\eta, \underline{\eta}). \end{aligned}$$

Since

$$|\eta - \underline{\eta}|^2 = |\eta|^2 + |\underline{\eta}|^2 - 2(\eta, \underline{\eta}),$$

we obtain the sum identity

$$D\underline{\omega} + \underline{D}\omega = \Omega \left(4\omega\underline{\omega} - \rho_{\mathbf{Rm}} + \frac{1}{2}|\eta|^2 + \frac{1}{2}|\underline{\eta}|^2 - 2(\eta, \underline{\eta}) \right).$$

Next we derive the difference identity from the scalar commutator applied to $\log \Omega$. Since

$$[L, \underline{L}] = -4\Omega^2 \zeta^\#,$$

we have

$$\begin{aligned} D(\underline{D} \log \Omega) - \underline{D}(D \log \Omega) &= [L, \underline{L}] \log \Omega \\ &= -4\Omega^2 (\zeta, \not{d} \log \Omega). \end{aligned}$$

Using

$$\zeta = \frac{1}{2}(\eta - \underline{\eta}), \qquad \not{d} \log \Omega = \frac{1}{2}(\eta + \underline{\eta}),$$

this becomes

$$D(\underline{D} \log \Omega) - \underline{D}(D \log \Omega) = -\Omega^2 (|\eta|^2 - |\underline{\eta}|^2).$$

On the other hand,

$$D \log \Omega = -2\Omega\omega, \qquad \underline{D} \log \Omega = -2\Omega\underline{\omega}.$$

Hence

$$\begin{aligned} D(\underline{D} \log \Omega) - \underline{D}(D \log \Omega) &= D(-2\Omega\underline{\omega}) - \underline{D}(-2\Omega\omega) \\ &= -2D\Omega\, \underline{\omega} - 2\Omega D\underline{\omega} + 2\underline{D}\Omega\, \omega + 2\Omega \underline{D}\omega. \end{aligned}$$

Since

$$D\Omega = -2\Omega^2\omega, \qquad \underline{D}\Omega = -2\Omega^2\underline{\omega},$$

the quadratic terms cancel, and we get

$$D(\underline{D} \log \Omega) - \underline{D}(D \log \Omega) = -2\Omega(D\underline{\omega} - \underline{D}\omega).$$

Therefore

$$D\underline{\omega} - \underline{D}\omega = \frac{\Omega}{2} \left(|\eta|^2 - |\underline{\eta}|^2 \right).$$

Combining the sum and difference identities gives

$$\begin{aligned} D\underline{\omega} &= \frac{1}{2} \left[(D\underline{\omega} + \underline{D}\omega) + (D\underline{\omega} - \underline{D}\omega) \right] \\ &= \Omega \left(2\omega\underline{\omega} - (\eta, \underline{\eta}) + \frac{1}{2}|\eta|^2 - \frac{1}{2}\rho_{\mathbf{Rm}} \right), \end{aligned}$$

and

$$\begin{aligned} \underline{D}\omega &= \frac{1}{2} \left[(D\underline{\omega} + \underline{D}\omega) - (D\underline{\omega} - \underline{D}\omega) \right] \\ &= \Omega \left(2\omega\underline{\omega} - (\eta, \underline{\eta}) + \frac{1}{2}|\underline{\eta}|^2 - \frac{1}{2}\rho_{\mathbf{Rm}} \right). \end{aligned}$$

Finally, the same-direction identities follow only by differentiating the lapse relations. From

$$D \log \Omega = -2\Omega\omega,$$

we get

$$D^2 \log \Omega = -2D\Omega\, \omega - 2\Omega D\omega = 4\Omega^2\omega^2 - 2\Omega D\omega,$$

and hence

$$D\omega = 2\Omega\omega^2 - \frac{1}{2\Omega} D^2 \log \Omega.$$

The identity for $\underline{D}\underline{\omega}$ is identical. □

Remark 2.14. The same-direction equations $D\omega$, $\underline{D}\underline{\omega}$ are the lapse propagation equations. The mixed equations $D\underline{\omega}$, $\underline{D}\omega$ are obtained by combining the scalar commutator for $\log\Omega$ with the e_4, e_3, e_4, e_3 curvature identity.

3 Characteristic initial data

We describe the characteristic data for the vacuum Einstein equations on two intersecting null hypersurfaces

$$C_{u_0} \cup \underline{C}_{\underline{u}_0}, \qquad S_0 = C_{u_0} \cap \underline{C}_{\underline{u}_0}.$$

The outgoing hypersurface C_{u_0} is foliated by

$$S_{\underline{u}} = C_{u_0} \cap \underline{C}_{\underline{u}},$$

and the incoming hypersurface $\underline{C}_{\underline{u}_0}$ is foliated by

$$S_u = C_u \cap \underline{C}_{\underline{u}_0}.$$

3.1 Initial double-null gauge

On C_{u_0}, let L' be the affine generator:

$$\nabla_{L'} L' = 0.$$

Choose $\underline{u}$ on C_{u_0} by

$$L'\underline{u} = 1, \qquad \underline{u}|_{S_0} = \underline{u}_0.$$

On $\underline{C}_{\underline{u}_0}$, let $\underline{L}'$ be the affine generator:

$$\nabla_{\underline{L}'} \underline{L}' = 0.$$

Choose u on $\underline{C}_{\underline{u}_0}$ by

$$\underline{L}'u = 1, \qquad u|_{S_0} = u_0.$$

The optical functions $u, \underline{u}$ near the initial hypersurfaces are then obtained by solving

$$\mathbf{g}^{-1}(du, du) = 0, \qquad \mathbf{g}^{-1}(d\underline{u}, d\underline{u}) = 0.$$

With

$$L = \Omega e_4, \qquad \underline{L} = \Omega e_3,$$

the affine normalizations give

$$\Omega = 1 \qquad \text{on } C_{u_0} \cup \underline{C}_{\underline{u}_0}.$$

Hence, on C_{u_0},

$$L = L', \qquad e_4 = L', \qquad D = L', \qquad \omega = 0,$$

and, on $\underline{C}_{\underline{u}_0}$,

$$\underline{L} = \underline{L}', \qquad e_3 = \underline{L}', \qquad \underline{D} = \underline{L}', \qquad \underline{\omega} = 0.$$

Since $\Omega = 1$ on the initial hypersurfaces, its angular derivative vanishes there. Thus

$$\eta = \zeta, \qquad \underline{\eta} = -\zeta \qquad \text{on } C_{u_0} \cup \underline{C}_{\underline{u}_0}.$$

Choose angular coordinates ϑ^A on S_0. On C_{u_0}, transport them by

$$L\vartheta^A = 0.$$

On $\underline{C}_{\underline{u}_0}$, transport them by

$$\underline{L}\vartheta^A = 0.$$

In the double-null coordinates

$$\underline{L} = \partial_u, \qquad L = \partial_{\underline{u}} + b^A \partial_A,$$

the first transport condition gives

$$b^A = 0 \qquad \text{on } C_{u_0}.$$

There is no incoming shift in the convention $\underline{L} = \partial_u$. The shift away from C_{u_0} is determined by

$$\partial_u b^A = 4\Omega^2 \zeta^A, \qquad b^A|_{C_{u_0}} = 0.$$

3.2 Geometric quantities on the initial hypersurfaces

On each section $S_{u,\underline{u}}$, let

$$\not{g} = \mathbf{g}|_{S_{u,\underline{u}}}$$

be the induced metric. On C_{u_0}, the null direction e_4 is tangent to the hypersurface, while e_3 is transverse. Thus ∇_4 is an intrinsic transport direction on C_{u_0}, but ∇_3 is not. Conversely, on $\underline{C}_{\underline{u}_0}$, e_3 is tangent and e_4 is transverse.

On C_{u_0}, the outgoing second fundamental form is

$$\chi_{AB} = \mathbf{g}(\nabla_A e_4, e_B).$$

Since e_4 is tangent to C_{u_0},

$$D\not{g} = 2\chi.$$

Thus χ is determined by the variation of $\not{g}$ along C_{u_0}. By contrast,

$$\underline{\chi}_{AB} = \mathbf{g}(\nabla_A e_3, e_B)$$

is a transverse second fundamental form on C_{u_0}. It is defined after choosing the transverse null normal e_3, but it is not obtained by differentiating $\not{g}$ along C_{u_0}. Its components

$$\mathrm{tr}\underline{\chi}, \qquad \hat{\underline{\chi}}$$

are reconstructed from the constraint equations along C_{u_0}.

Similarly, on $\underline{C}_{\underline{u}_0}$, the incoming second fundamental form is

$$\underline{\chi}_{AB} = \mathbf{g}(\nabla_A e_3, e_B).$$

Since e_3 is tangent to $\underline{C}_{\underline{u}_0}$,

$$\underline{D}\not{g} = 2\underline{\chi}.$$

Thus $\underline{\chi}$ is determined by the variation of $\not{g}$ along $\underline{C}_{\underline{u}_0}$. By contrast,

$$\chi_{AB} = \mathbf{g}(\nabla_A e_4, e_B)$$

is transverse on $\underline{C}_{\underline{u}_0}$, and its components

$$\mathrm{tr}\chi, \qquad \hat{\chi}$$

are reconstructed from the conjugate constraint equations.

The normal connection one-form is

$$\zeta_A = \frac{1}{2}\mathbf{g}(\nabla_A e_4, e_3).$$

It is defined on both initial hypersurfaces once the pair of null normals (e_3, e_4) has been fixed. Therefore, on C_{u_0}, the transverse quantities to reconstruct are

$$\zeta, \qquad \mathrm{tr}\underline{\chi}, \qquad \hat{\underline{\chi}},$$

whereas on $\underline{C}_{\underline{u}_0}$, they are

$$\zeta, \qquad \mathrm{tr}\chi, \qquad \hat{\chi}.$$

The characteristic constraints are precisely the transport equations which recover these transverse quantities from their values on the corner S_0.

3.3 Constraint equations on the initial hypersurfaces

The constraints on C_{u_0} are the null structure equations involving only D-derivatives, angular derivatives on $S_{\underline{u}}$, and quantities defined on C_{u_0}. In the initial gauge, they reduce to

$$D\not{g} = 2\chi,$$

$$D\mathrm{tr}\chi = -\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2,$$

$$D\zeta + (\mathrm{tr}\chi)\zeta = \not{\mathrm{d}}\mathrm{iv}\chi - \not{d}\mathrm{tr}\chi,$$

$$D\mathrm{tr}\underline{\chi} + (\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) = 2(-K - \not{\mathrm{d}}\mathrm{iv}\zeta + |\zeta|^2),$$

and

$$D\underline{\hat{\chi}} - (\hat{\chi}, \underline{\hat{\chi}})\not{g} - \frac{1}{2}(\mathrm{tr}\chi)\underline{\hat{\chi}} = -\not{\nabla}\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \frac{1}{2}(\mathrm{tr}\underline{\chi})\hat{\chi}.$$

Here

$$(\not{\nabla}\widehat{\otimes}\zeta)_{AB} = \not{\nabla}_A\zeta_B + \not{\nabla}_B\zeta_A - (\not{\mathrm{d}}\mathrm{iv}\zeta)\not{g}_{AB},$$

and

$$(\zeta\widehat{\otimes}\zeta)_{AB} = \zeta_A\zeta_B - \frac{1}{2}|\zeta|^2\not{g}_{AB}.$$

The constraints on $\underline{C}_{\underline{u}_0}$ are the conjugate hierarchy. Since interchanging e_4 and e_3 sends

$$\zeta = \frac{1}{2}\mathbf{g}(\nabla_A e_4, e_3) \qquad \text{to} \qquad -\zeta,$$

one obtains

$$\underline{D}\not{g} = 2\underline{\chi},$$

$$\underline{D}\mathrm{tr}\underline{\chi} = -\frac{1}{2}(\mathrm{tr}\underline{\chi})^2 - |\underline{\hat{\chi}}|^2,$$

$$\underline{D}\zeta + (\mathrm{tr}\underline{\chi})\zeta = -\not{\mathrm{d}}\mathrm{iv}\underline{\chi} + \not{d}\mathrm{tr}\underline{\chi},$$

$$\underline{D}\mathrm{tr}\chi + (\mathrm{tr}\underline{\chi})(\mathrm{tr}\chi) = 2(-K + \not{\mathrm{d}}\mathrm{iv}\zeta + |\zeta|^2),$$

and

$$\underline{D}\hat{\chi} - (\underline{\hat{\chi}}, \hat{\chi})\not{g} - \frac{1}{2}(\mathrm{tr}\underline{\chi})\hat{\chi} = \not{\nabla}\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \frac{1}{2}(\mathrm{tr}\chi)\underline{\hat{\chi}}.$$

3.4 Scale, conformal data, and shear

The intrinsic free data on each initial null hypersurface is the conformal class of the induced metrics on the sections.

On C_{u_0}, identify the sections $S_{\underline{u}}$ with S_0 by the transported angular coordinates. Write

$$\not{g}(\underline{u}) = \phi_+^2(\underline{u})\widehat{\not{g}}_+(\underline{u}), \qquad d\mu_{\widehat{\not{g}}_+(\underline{u})} = d\mu_{\not{g}_0},$$

where

$$\not{g}_0 = \not{g}|_{S_0}.$$

Then

$$\mathrm{tr}\chi = 2\frac{D\phi_+}{\phi_+}, \qquad \hat{\chi}_{AB} = \frac{1}{2}\phi_+^2 D(\widehat{\not{g}}_+)_{AB}.$$

If

$$e_+ = \frac{1}{2}|\hat{\chi}|^2_{\not{g}} = \frac{1}{8}\widehat{\not{g}}_+^{AC}\widehat{\not{g}}_+^{BD} D(\widehat{\not{g}}_+)_{AB} D(\widehat{\not{g}}_+)_{CD},$$

then Raychaudhuri is equivalent to

$$D^2\phi_+ + e_+\phi_+ = 0.$$

Similarly, on $\underline{C}_{\underline{u}_0}$, identify the sections S_u with S_0 and write

$$\not{g}(u) = \phi_-^2(u)\widehat{\not{g}}_-(u), \qquad d\mu_{\widehat{\not{g}}_-(u)} = d\mu_{\not{g}_0}.$$

Then

$$\mathrm{tr}\underline{\chi} = 2\frac{\underline{D}\phi_-}{\phi_-}, \qquad \hat{\underline{\chi}}_{AB} = \frac{1}{2}\phi_-^2\underline{D}(\widehat{\not{g}}_-)_{AB},$$

and

$$\underline{D}^2\phi_- + e_-\phi_- = 0,$$

where

$$e_- = \frac{1}{2}|\hat{\underline{\chi}}|^2_{\not{g}} = \frac{1}{8}\widehat{\not{g}}_-^{AC}\widehat{\not{g}}_-^{BD}\underline{D}(\widehat{\not{g}}_-)_{AB}\underline{D}(\widehat{\not{g}}_-)_{CD}.$$

Thus the shear data may be prescribed either as the fixed-area conformal metric curves

$$\widehat{\not{g}}_+(\underline{u}), \qquad \widehat{\not{g}}_-(u),$$

or equivalently as

$$\hat{\chi} \quad \text{on } C_{u_0}, \qquad \hat{\underline{\chi}} \quad \text{on } \underline{C}_{\underline{u}_0}.$$

The two descriptions are equivalent after solving the Raychaudhuri equations for the scale factors ϕ_+ and ϕ_-.

3.5 Free data and reconstruction

In the gauge above, the free characteristic data consist of the following.

1. The corner metric

$$\not{g}_0 \qquad \text{on } S_0.$$

2. Two fixed-area conformal metric curves

$$\widehat{\not{g}}_+(\underline{u}), \qquad \widehat{\not{g}}_-(u),$$

   satisfying

$$\widehat{\not{g}}_+(\underline{u}_0) = \widehat{\not{g}}_-(u_0) = \not{g}_0,$$

   and

$$d\mu_{\widehat{\not{g}}_+(\underline{u})} = d\mu_{\widehat{\not{g}}_-(u)} = d\mu_{\not{g}_0}.$$

3. The corner transverse data

$$\zeta_0, \qquad (\mathrm{tr}\chi)_0, \qquad (\mathrm{tr}\underline{\chi})_0.$$

The scale factors are determined by

$$D^2\phi_+ + e_+\phi_+ = 0, \qquad \phi_+|_{S_0} = 1, \qquad D\phi_+|_{S_0} = \frac{1}{2}(\mathrm{tr}\chi)_0,$$

and

$$\underline{D}^2\phi_- + e_-\phi_- = 0, \qquad \phi_-|_{S_0} = 1, \qquad \underline{D}\phi_-|_{S_0} = \frac{1}{2}(\mathrm{tr}\underline{\chi})_0.$$

Thus the initial scale derivatives are not extra data; they are the corner expansions.

Theorem 3.1 (Reconstruction from double characteristic data)**.** *Assume the scale factors ϕ_+ and ϕ_- solving the two equations above remain positive. Then the characteristic constraints uniquely determine the connection coefficients on*

$$C_{u_0} \cup \underline{C}_{\underline{u}_0}$$

which enter the initial constraint system.

Proof. On C_{u_0}, the conformal curve $\widehat{\not{g}}_+$ and the scale ϕ_+ determine

$$\not{g} = \phi_+^2 \widehat{\not{g}}_+, \qquad \mathrm{tr}\chi = 2\frac{D\phi_+}{\phi_+}, \qquad \hat{\chi}_{AB} = \frac{1}{2}\phi_+^2 D(\widehat{\not{g}}_+)_{AB}.$$

Hence χ is known. Starting from the corner values

$$\zeta|_{S_0} = \zeta_0, \qquad \mathrm{tr}\underline{\chi}|_{S_0} = (\mathrm{tr}\underline{\chi})_0, \qquad \hat{\underline{\chi}}|_{S_0} = \frac{1}{2}\underline{D}(\widehat{\not{g}}_-)_{AB}\big|_{S_0},$$

one solves successively

$$D\zeta + (\mathrm{tr}\chi)\zeta = \mathrm{d}\not{\mathrm{i}}\mathrm{v}\chi - \not{d}\mathrm{tr}\chi,$$

$$D\mathrm{tr}\underline{\chi} + (\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) = 2(-K - \mathrm{d}\not{\mathrm{i}}\mathrm{v}\zeta + |\zeta|^2),$$

and

$$D\hat{\underline{\chi}} - (\hat{\chi}, \hat{\underline{\chi}})\not{g} - \frac{1}{2}(\mathrm{tr}\chi)\hat{\underline{\chi}} = -\not{\nabla}\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \frac{1}{2}(\mathrm{tr}\underline{\chi})\hat{\chi}.$$

Each equation is an ODE along the generators of C_{u_0}.

On $\underline{C}_{\underline{u}_0}$, the conformal curve $\widehat{\not{g}}_-$ and the scale ϕ_- determine

$$\not{g} = \phi_-^2 \widehat{\not{g}}_-, \qquad \mathrm{tr}\underline{\chi} = 2\frac{\underline{D}\phi_-}{\phi_-}, \qquad \hat{\underline{\chi}}_{AB} = \frac{1}{2}\phi_-^2 \underline{D}(\widehat{\not{g}}_-)_{AB}.$$

Hence $\underline{\chi}$ is known. Starting from

$$\zeta|_{S_0} = \zeta_0, \qquad \mathrm{tr}\chi|_{S_0} = (\mathrm{tr}\chi)_0, \qquad \hat{\chi}|_{S_0} = \frac{1}{2}D(\widehat{\not{g}}_+)_{AB}\big|_{S_0},$$

one solves successively

$$\underline{D}\zeta + (\mathrm{tr}\underline{\chi})\zeta = -\mathrm{d}\not{\mathrm{i}}\mathrm{v}\underline{\chi} + \not{d}\mathrm{tr}\underline{\chi},$$

$$\underline{D}\mathrm{tr}\chi + (\mathrm{tr}\underline{\chi})(\mathrm{tr}\chi) = 2(-K + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\zeta + |\zeta|^2),$$

and

$$\underline{D}\hat{\chi} - (\hat{\underline{\chi}}, \hat{\chi})\not{g} - \frac{1}{2}(\mathrm{tr}\underline{\chi})\hat{\chi} = \not{\nabla}\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \frac{1}{2}(\mathrm{tr}\chi)\hat{\underline{\chi}}.$$

Thus the initial connection coefficients are reconstructed on both initial null hypersurfaces. □

Remark 3.2 (Functional degrees of freedom). The main free data are the two fixed-area conformal metric curves

$$\widehat{\not{g}}_+(\underline{u}), \qquad \widehat{\not{g}}_-(u).$$

A metric on a two-dimensional section has three components, and fixing the area form removes one. Thus each curve carries two scalar functions on the corresponding null hypersurface. These are the two radiative degrees of freedom of the gravitational field.

The remaining free data

$$\not{g}_0, \qquad \zeta_0, \qquad (\mathrm{tr}\chi)_0, \qquad (\mathrm{tr}\underline{\chi})_0$$

live only on the corner S_0. They fix the initial metric, normal connection, and two initial expansions. They are lower-dimensional corner data, not additional radiative data along the null hypersurfaces.

3.6 Curvature induced on the initial hypersurfaces

After the connection coefficients are reconstructed, the curvature components on C_{u_0} are recovered from the null structure equations:

$$\alpha = -D\hat{\chi} - (\mathrm{tr}\chi)\hat{\chi},$$

$$\beta = -\mathrm{d}\!\!/\mathrm{iv}\chi + \mathrm{d}\!\!/\mathrm{tr}\chi - \chi\cdot\zeta + (\mathrm{tr}\chi)\zeta,$$

$$\rho = -K - \frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) + \frac{1}{2}(\hat{\chi}, \hat{\underline{\chi}}),$$

and

$$\sigma = \frac{1}{2}\chi\wedge\underline{\chi} - \mathrm{c}\!\!/\mathrm{url}\zeta.$$

The remaining curvature components on C_{u_0} are obtained from the Bianchi equations along C_{u_0}.

Similarly, on $\underline{C}_{\underline{u}_0}$, the structure equations recover

$$\underline{\alpha} = -\underline{D}\hat{\underline{\chi}} - (\mathrm{tr}\underline{\chi})\hat{\underline{\chi}},$$

$$\underline{\beta} = \mathrm{d}\!\!/\mathrm{iv}\underline{\chi} - \mathrm{d}\!\!/\mathrm{tr}\underline{\chi} - \underline{\chi}\cdot\zeta + (\mathrm{tr}\underline{\chi})\zeta,$$

$$\rho = -K - \frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) + \frac{1}{2}(\hat{\chi}, \hat{\underline{\chi}}),$$

and

$$\sigma = \frac{1}{2}\chi\wedge\underline{\chi} - \mathrm{c}\!\!/\mathrm{url}\zeta.$$

The remaining curvature components on $\underline{C}_{\underline{u}_0}$ are obtained from the Bianchi equations along $\underline{C}_{\underline{u}_0}$.

4 Bianchi equations

We use the curvature convention

$$\mathbf{Rm}(X, Y, Z, W) = \mathbf{g}\big(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\big).$$

In vacuum, the Weyl tensor satisfies the contracted Bianchi equation

$$\nabla^\alpha \mathbf{W}_{\alpha\beta\gamma\delta} = 0.$$

Relative to the normalized null frame (e_3, e_4, e_A), with

$$\mathbf{g}(e_3, e_4) = -2,$$

we use

$$\alpha_{AB} = \mathbf{W}_{A4B4}, \qquad \beta_A = \frac{1}{2}\mathbf{W}_{A434},$$

$$\rho = \frac{1}{4}\mathbf{W}_{3434}, \qquad \sigma = \frac{1}{4}\epsilon\!\!/^{AB}\mathbf{W}_{AB34},$$

$$\underline{\beta}_A = \frac{1}{2}\mathbf{W}_{A334}, \qquad \underline{\alpha}_{AB} = \mathbf{W}_{A3B3}.$$

We write schematically

$$\mathbf{W} \in \{\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}, \qquad \Gamma \in \{\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \omega, \underline{\omega}\}.$$

For an S-tensorfield Ψ, set

$$\nabla\!\!\!\!/_4\Psi = (\nabla_{e_4}\Psi)^\top, \qquad \nabla\!\!\!\!/_3\Psi = (\nabla_{e_3}\Psi)^\top.$$

4.1 Maxwell form and the Bel–Robinson tensor

The invariant Maxwell form of the Bianchi equations is the divergence equation for the Weyl field:

$$\mathrm{div}\,\mathbf{W} = 0, \qquad (\mathrm{div}\,\mathbf{W})_{\beta\gamma\delta} = \nabla^\alpha \mathbf{W}_{\alpha\beta\gamma\delta}.$$

This is the spin 2 analogue of the source-free Maxwell equation

$$\nabla^\alpha F_{\alpha\beta} = 0, \qquad \nabla^{\alpha\,*} F_{\alpha\beta} = 0.$$

The associated stress-energy tensor is the Bel–Robinson tensor

$$Q[\mathbf{W}]_{\alpha\beta\gamma\delta} = \mathbf{W}_{\alpha\mu\gamma\nu}\mathbf{W}_\beta{}^\mu{}_\delta{}^\nu + {}^*\mathbf{W}_{\alpha\mu\gamma\nu}{}^*\mathbf{W}_\beta{}^\mu{}_\delta{}^\nu.$$

In vacuum,

$$\nabla^\alpha Q[\mathbf{W}]_{\alpha\beta\gamma\delta} = 0.$$

Thus, for vectorfields X, Y, Z, the Bel–Robinson current

$$P_\alpha[\mathbf{W}; X, Y, Z] = Q[\mathbf{W}]_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta$$

satisfies

$$\begin{aligned}\nabla^\alpha P_\alpha[\mathbf{W}; X, Y, Z] = Q[\mathbf{W}] \cdot {}^{(X)}\pi \cdot Y \cdot Z + Q[\mathbf{W}] \cdot X \cdot {}^{(Y)}\pi \cdot Z \\ + Q[\mathbf{W}] \cdot X \cdot Y \cdot {}^{(Z)}\pi,\end{aligned}$$

where

$${}^{(X)}\pi = \mathcal{L}_X \mathbf{g}$$

is the deformation tensor. Hence if X, Y, Z are Killing, the current is divergence-free.

The null components of $Q[\mathbf{W}]$ are positive quadratic expressions in the Weyl components:

$$Q_{4444} \sim |\alpha|^2, \qquad Q_{3444} \sim |\beta|^2, \qquad Q_{3344} \sim \rho^2 + \sigma^2,$$

$$Q_{3334} \sim |\underline{\beta}|^2, \qquad Q_{3333} \sim |\underline{\alpha}|^2.$$

This is the Christodoulou–Klainerman energy mechanism: the invariant equation $\mathrm{div}\,\mathbf{W} = 0$ produces divergence identities for $Q[\mathbf{W}]$, and the null decomposition of $Q[\mathbf{W}]$ gives coercive fluxes for

$$\alpha, \quad \beta, \quad \rho, \sigma, \quad \underline{\beta}, \quad \underline{\alpha}.$$

4.2 Hodge packages on $S_{u,\underline{u}}$

The null decomposition of $\mathrm{div}\,\mathbf{W} = 0$ gives a chain of first-order equations on the sections $S_{u,\underline{u}}$. The angular principal operators are the standard Hodge operators on $S_{u,\underline{u}}$.

Set

$$\mathfrak{s}_0 = C^\infty(S) \oplus C^\infty(S), \qquad \mathfrak{s}_1 = \Omega^1(S), \qquad \mathfrak{s}_2 = \{\text{symmetric trace-free } S\text{-two-tensors}\}.$$

We use

$$(-\rho, \sigma) \in \mathfrak{s}_0 \qquad \text{for the } \beta\text{-pair},$$

and

$$(\rho, \sigma) \in \mathfrak{s}_0 \qquad \text{for the } \underline{\beta}\text{-pair}.$$

Define

$$\not{d}_1 : \mathfrak{s}_1 \to \mathfrak{s}_0, \qquad \not{d}_1 \xi = (\not{\mathrm{div}}\,\xi, \not{\mathrm{curl}}\,\xi),$$

and

$$\not{d}_1^* : \mathfrak{s}_0 \to \mathfrak{s}_1, \qquad \not{d}_1^*(f, g) = -\not{d} f + {}^*\not{d} g,$$

where

$$\operatorname{d\!\!/iv}\xi = \nabla\!\!\!/^A \xi_A, \qquad \operatorname{c\!\!/url}\xi = \in\!\!\!/^{AB}\nabla\!\!\!/_A\xi_B, \qquad ({}^*d\!\!\!/ g)_A = \in\!\!\!/_A{}^B\nabla\!\!\!/_B g.$$

Thus

$$d\!\!\!/_1^*(-\rho,\sigma) = d\!\!\!/\rho + {}^*d\!\!\!/\sigma, \qquad d\!\!\!/_1^*(\rho,\sigma) = -d\!\!\!/\rho + {}^*d\!\!\!/\sigma.$$

For symmetric trace-free two-tensors,

$$d\!\!\!/_2 : \mathfrak{s}_2 \to \mathfrak{s}_1, \qquad (d\!\!\!/_2\theta)_A = \nabla\!\!\!/^B\theta_{AB},$$

and

$$d\!\!\!/_2^* : \mathfrak{s}_1 \to \mathfrak{s}_2, \qquad (d\!\!\!/_2^*\xi)_{AB} = -\frac{1}{2}\left(\nabla\!\!\!/_A\xi_B + \nabla\!\!\!/_B\xi_A - (\operatorname{d\!\!/iv}\xi)g\!\!\!/_{AB}\right).$$

On a closed section $S_{u,\underline{u}}$,

$$\int_S \langle d\!\!\!/_1\xi, (f,g)\rangle \, d\mu_{g\!\!\!/} = \int_S \langle \xi, d\!\!\!/_1^*(f,g)\rangle \, d\mu_{g\!\!\!/},$$

and

$$\int_S \langle d\!\!\!/_2\theta, \xi\rangle \, d\mu_{g\!\!\!/} = \int_S \langle \theta, d\!\!\!/_2^*\xi\rangle \, d\mu_{g\!\!\!/}.$$

Sketch of proof. For $d\!\!\!/_1$, integrate by parts on the closed surface S:

$$\begin{aligned}\int_S \langle d\!\!\!/_1\xi, (f,g)\rangle \, d\mu_{g\!\!\!/} &= \int_S \left(f\,\nabla\!\!\!/^A\xi_A + g\,\in\!\!\!/^{AB}\nabla\!\!\!/_A\xi_B\right) d\mu_{g\!\!\!/} \\ &= -\int_S \left(\xi^A\nabla\!\!\!/_A f + \xi_B\in\!\!\!/^{AB}\nabla\!\!\!/_A g\right) d\mu_{g\!\!\!/} \\ &= \int_S \xi^A\left(-\nabla\!\!\!/_A f + \in\!\!\!/_A{}^B\nabla\!\!\!/_B g\right) d\mu_{g\!\!\!/} \\ &= \int_S \langle \xi, d\!\!\!/_1^*(f,g)\rangle \, d\mu_{g\!\!\!/}.\end{aligned}$$

Here we used

$$\nabla\!\!\!/\in\!\!\!/ = 0$$

and the absence of boundary terms.

For $d\!\!\!/_2$, since θ is symmetric trace-free,

$$\begin{aligned}\int_S \langle d\!\!\!/_2\theta, \xi\rangle \, d\mu_{g\!\!\!/} &= \int_S \xi^A\nabla\!\!\!/^B\theta_{AB}\, d\mu_{g\!\!\!/} \\ &= -\int_S \theta_{AB}\nabla\!\!\!/^B\xi^A\, d\mu_{g\!\!\!/} \\ &= -\frac{1}{2}\int_S \theta^{AB}\left(\nabla\!\!\!/_A\xi_B + \nabla\!\!\!/_B\xi_A\right) d\mu_{g\!\!\!/} \\ &= -\frac{1}{2}\int_S \theta^{AB}\left(\nabla\!\!\!/_A\xi_B + \nabla\!\!\!/_B\xi_A - (\operatorname{d\!\!/iv}\xi)g\!\!\!/_{AB}\right) d\mu_{g\!\!\!/} \\ &= \int_S \langle \theta, d\!\!\!/_2^*\xi\rangle \, d\mu_{g\!\!\!/}.\end{aligned}$$

The insertion of

$$-(\operatorname{d\!\!/iv}\xi)g\!\!\!/_{AB}$$

does not change the integral because

$$g\!\!\!/^{AB}\theta_{AB} = 0.$$

Thus $d\!\!\!/_1^*$ and $d\!\!\!/_2^*$ are the formal $L^2(S)$-adjoints of $d\!\!\!/_1$ and $d\!\!\!/_2$, respectively. □

4.3 Schematic null Bianchi system

The Weyl components form the Hodge chain

$$\alpha \longleftrightarrow \beta \longleftrightarrow (-\rho, \sigma) \longleftrightarrow \underline{\beta} \longleftrightarrow \underline{\alpha}.$$

Proposition 4.1 (Principal null Bianchi system)**.** *Suppressing all lower-order Ricci-coefficient terms into* $\Gamma \cdot \mathbf{W}$*, the vacuum Bianchi equations imply*

$$\begin{aligned} \nabla\!\!\!/_3\alpha + d\!\!\!/_2^*\beta &= \Gamma \cdot \mathbf{W}, & \nabla\!\!\!/_4\beta - d\!\!\!/_2\alpha &= \Gamma \cdot \mathbf{W}, \\ \nabla\!\!\!/_3\beta - d\!\!\!/_1^*(-\rho, \sigma) &= \Gamma \cdot \mathbf{W}, & \nabla\!\!\!/_4(-\rho, \sigma) + d\!\!\!/_1\beta &= \Gamma \cdot \mathbf{W}, \\ \nabla\!\!\!/_4\underline{\beta} + d\!\!\!/_1^*(\rho, \sigma) &= \Gamma \cdot \mathbf{W}, & \nabla\!\!\!/_3(\rho, \sigma) - d\!\!\!/_1\underline{\beta} &= \Gamma \cdot \mathbf{W}, \\ \nabla\!\!\!/_4\underline{\alpha} + d\!\!\!/_2^*\underline{\beta} &= \Gamma \cdot \mathbf{W}, & \nabla\!\!\!/_3\underline{\beta} - d\!\!\!/_2\underline{\alpha} &= \Gamma \cdot \mathbf{W}. \end{aligned}$$

In particular, the middle scalar equations are

$$\nabla\!\!\!/_4\rho = \mathrm{d\!\!\!/iv}\beta + \Gamma \cdot \mathbf{W}, \qquad \nabla\!\!\!/_4\sigma = -\mathrm{cu\!\!\!/rl}\beta + \Gamma \cdot \mathbf{W},$$

and

$$\nabla\!\!\!/_3\rho = \mathrm{d\!\!\!/iv}\underline{\beta} + \Gamma \cdot \mathbf{W}, \qquad \nabla\!\!\!/_3\sigma = \mathrm{cu\!\!\!/rl}\underline{\beta} + \Gamma \cdot \mathbf{W}.$$

Proof. The contracted Bianchi equation is

$$\nabla^\alpha \mathbf{W}_{\alpha\beta\gamma\delta} = 0.$$

Since

$$\mathbf{g}^{-1} = -\frac{1}{2}(e_3 \otimes e_4 + e_4 \otimes e_3) + g\!\!\!/^{AB} e_A \otimes e_B,$$

its principal part is

$$-\frac{1}{2}\nabla_4 \mathbf{W}_{3\beta\gamma\delta} - \frac{1}{2}\nabla_3 \mathbf{W}_{4\beta\gamma\delta} + \nabla\!\!\!/^A \mathbf{W}_{A\beta\gamma\delta} = \Gamma \cdot \mathbf{W}.$$

All derivatives of the frame, of $g\!\!\!/$, and of $\in\!\!\!\!/$ are absorbed into $\Gamma \cdot \mathbf{W}$.

For the end pair (α, β), the symmetric trace-free angular projection gives

$$\nabla\!\!\!/_3\alpha + d\!\!\!/_2^*\beta = \Gamma \cdot \mathbf{W},$$

and the one-form projection gives

$$\nabla\!\!\!/_4\beta - d\!\!\!/_2\alpha = \Gamma \cdot \mathbf{W}.$$

Indeed the angular derivative of β contributes through the trace-free symmetrized gradient $d\!\!\!/_2^*\beta$, while the angular derivative of α contributes through the divergence $d\!\!\!/_2\alpha$. Exchanging 3 and 4 gives

$$\nabla\!\!\!/_4\underline{\alpha} + d\!\!\!/_2^*\underline{\beta} = \Gamma \cdot \mathbf{W}, \qquad \nabla\!\!\!/_3\underline{\beta} - d\!\!\!/_2\underline{\alpha} = \Gamma \cdot \mathbf{W}.$$

It remains to record the middle signs. Using

$$\mathbf{W}_{A434} = 2\beta_A, \qquad \mathbf{W}_{A334} = 2\underline{\beta}_A, \qquad \mathbf{W}_{3434} = 4\rho, \qquad \mathbf{W}_{AB34} = 2\sigma \in\!\!\!\!/_{AB},$$

the outgoing scalar projections give

$$\nabla\!\!\!/_4\rho = \mathrm{d\!\!\!/iv}\beta + \Gamma \cdot \mathbf{W}, \qquad \nabla\!\!\!/_4\sigma = -\mathrm{cu\!\!\!/rl}\beta + \Gamma \cdot \mathbf{W}.$$

Therefore

$$\nabla\!\!\!/_4(-\rho, \sigma) + d\!\!\!/_1\beta = \Gamma \cdot \mathbf{W},$$

because

$$d\!\!\!/_1\beta = (\mathrm{d\!\!\!/iv}\beta, \mathrm{cu\!\!\!/rl}\beta).$$

The companion one-form projection gives

$$\nabla\!\!\!/_3\beta - d\!\!/\rho - {}^*d\!\!/\sigma = \Gamma\cdot\mathbf{W}.$$

Since

$$d\!\!/_1^*(-\rho,\sigma) = d\!\!/\rho + {}^*d\!\!/\sigma,$$

this is

$$\nabla\!\!\!/_3\beta - d\!\!/_1^*(-\rho,\sigma) = \Gamma\cdot\mathbf{W}.$$

Similarly, the incoming scalar projections give

$$\nabla\!\!\!/_3\rho = \mathrm{d\!\!/iv}\,\underline{\beta} + \Gamma\cdot\mathbf{W}, \qquad \nabla\!\!\!/_3\sigma = \mathrm{cu\!\!/rl}\,\underline{\beta} + \Gamma\cdot\mathbf{W},$$

or

$$\nabla\!\!\!/_3(\rho,\sigma) - d\!\!/_1\underline{\beta} = \Gamma\cdot\mathbf{W}.$$

The companion one-form projection gives

$$\nabla\!\!\!/_4\underline{\beta} - d\!\!/\rho + {}^*d\!\!/\sigma = \Gamma\cdot\mathbf{W}.$$

Since

$$d\!\!/_1^*(\rho,\sigma) = -d\!\!/\rho + {}^*d\!\!/\sigma,$$

this is

$$\nabla\!\!\!/_4\underline{\beta} + d\!\!/_1^*(\rho,\sigma) = \Gamma\cdot\mathbf{W}.$$

Combining the end pairs and middle pairs gives the stated Hodge chain. □

4.4 Component pairing and hidden divergence structure

The Hodge-chain form of the Bianchi equations yields a component-level energy identity. This is not the invariant Bel–Robinson identity itself; rather, it is the null-component version of the same cancellation mechanism.

Lemma 4.2 (Hodge cancellation)**.** *Let*

$$d\!\!/ : \mathfrak{s}_{i+1} \to \mathfrak{s}_i, \qquad d\!\!/^* : \mathfrak{s}_i \to \mathfrak{s}_{i+1}$$

be either $(d\!\!/_1, d\!\!/_1^*)$ *or* $(d\!\!/_2, d\!\!/_2^*)$*. Then there is a bilinear* S*-vectorfield* $\mathcal{P}(\mathbf{W}_1,\mathbf{W}_2)$ *such that*

$$2\langle d\!\!/\mathbf{W}_1, \mathbf{W}_2\rangle - 2\langle \mathbf{W}_1, d\!\!/^*\mathbf{W}_2\rangle = \mathrm{d\!\!/iv}\,\mathcal{P}(\mathbf{W}_1,\mathbf{W}_2).$$

In particular, on a closed section S*,*

$$\int_S \Big(\langle d\!\!/\mathbf{W}_1, \mathbf{W}_2\rangle - \langle \mathbf{W}_1, d\!\!/^*\mathbf{W}_2\rangle\Big)\, d\mu_{g\!\!/} = 0.$$

Proof. This is the pointwise form of the formal adjointness of the Hodge operators. For example, for $d\!\!/_1$,

$$\begin{aligned}
&\langle d\!\!/_1\xi, (f,g)\rangle - \langle \xi, d\!\!/_1^*(f,g)\rangle \\
&= f\,\nabla\!\!\!/^A\xi_A + g\,\epsilon\!\!/^{AB}\nabla\!\!\!/_A\xi_B - \xi^A(-\nabla\!\!\!/_A f + \epsilon\!\!/_A{}^B\nabla\!\!\!/_B g) \\
&= \nabla\!\!\!/^A(f\xi_A) + \nabla\!\!\!/_A(g\epsilon\!\!/^{AB}\xi_B).
\end{aligned}$$

Thus it is an S-divergence. The $d\!\!/_2$-identity is the same integration-by-parts identity for divergence and trace-free symmetrized gradient. □

Proposition 4.3 (Divergence identity for one Bianchi pair)**.** *Let $s = \pm 1$. Suppose*

$$\nabla\!\!\!/_3 \mathbf{W}_1 + s \not{d}^* \mathbf{W}_2 = \Gamma \cdot \mathbf{W}, \qquad \nabla\!\!\!/_4 \mathbf{W}_2 - s \not{d} \mathbf{W}_1 = \Gamma \cdot \mathbf{W}.$$

Then

$$\mathrm{div}_{\mathbf{g}} \left(|\mathbf{W}_1|^2 e_3 + |\mathbf{W}_2|^2 e_4 \right) = s\, \mathrm{d}\!\!\!/\mathrm{iv} \mathcal{P}(\mathbf{W}_1, \mathbf{W}_2) + \Gamma \cdot \mathbf{W} \cdot \mathbf{W}.$$

The same statement holds with e_3 and e_4 interchanged.

Proof. The principal part is

$$\begin{aligned} &2\langle \nabla\!\!\!/_3 \mathbf{W}_1, \mathbf{W}_1 \rangle + 2\langle \nabla\!\!\!/_4 \mathbf{W}_2, \mathbf{W}_2 \rangle \\ &= -2s\langle \not{d}^* \mathbf{W}_2, \mathbf{W}_1 \rangle + 2s\langle \not{d} \mathbf{W}_1, \mathbf{W}_2 \rangle + \Gamma \cdot \mathbf{W} \cdot \mathbf{W} \\ &= s\, \mathrm{d}\!\!\!/\mathrm{iv} \mathcal{P}(\mathbf{W}_1, \mathbf{W}_2) + \Gamma \cdot \mathbf{W} \cdot \mathbf{W}. \end{aligned}$$

The terms coming from differentiating e_3, e_4, the S-metric, and the volume form are lower order and are absorbed into $\Gamma \cdot \mathbf{W} \cdot \mathbf{W}$. □

Applying this to the four adjacent Bianchi pairs gives the schematic currents

$$\begin{aligned} &|\alpha|^2 e_3 + |\beta|^2 e_4, \qquad |\beta|^2 e_3 + (\rho^2 + \sigma^2) e_4, \\ &|\underline{\beta}|^2 e_4 + (\rho^2 + \sigma^2) e_3, \qquad |\underline{\alpha}|^2 e_4 + |\underline{\beta}|^2 e_3. \end{aligned}$$

4.5 Integrated component energy identity

Let

$$\mathcal{D}_{u_1, u_2; \underline{u}_1, \underline{u}_2} = \{ u_1 \le u \le u_2,\ \underline{u}_1 \le \underline{u} \le \underline{u}_2 \}.$$

Integrating the component divergence identity over $\mathcal{D}_{u_1, u_2; \underline{u}_1, \underline{u}_2}$ gives, for each adjacent Bianchi pair,

$$\text{future null fluxes} = \text{initial null fluxes} + \int_{\mathcal{D}_{u_1, u_2; \underline{u}_1, \underline{u}_2}} \Gamma \cdot \mathbf{W} \cdot \mathbf{W}.$$

The angular principal terms do not contribute because $S_{u,\underline{u}}$ is closed:

$$\int_{S_{u,\underline{u}}} \mathrm{d}\!\!\!/\mathrm{iv} \mathcal{P}(\mathbf{W}_1, \mathbf{W}_2)\, d\mu_{\not{g}} = 0.$$

Remark 4.4 (Comparison with the Bel–Robinson method)*.* The component pairing above is the modern null-decomposed way to see the same cancellation that is invariantly encoded by the Bel–Robinson tensor. The Christodoulou–Klainerman method uses the geometric current

$$P_\alpha[\mathbf{W}; X, Y, Z] = Q[\mathbf{W}]_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta,$$

and integrates its spacetime divergence. The component method instead pairs adjacent null Bianchi equations and uses the formal adjointness of the Hodge operators on $S_{u,\underline{u}}$. The former is invariant and multiplier-based; the latter is componentwise and displays explicitly where the angular derivatives cancel. Both express the same underlying fact: the Bianchi equations have a hidden divergence structure, and the curvature fluxes are positive quadratic forms in

$$\alpha, \quad \beta, \quad \rho, \sigma, \quad \underline{\beta}, \quad \underline{\alpha}.$$

5 Schematic mechanism of Luk's proof

Set

$$\Gamma := \{ \mathrm{tr}\chi, \hat{\chi}, \mathrm{tr}\underline{\chi}, \hat{\underline{\chi}}, \zeta, \eta, \underline{\eta}, \omega, \underline{\omega} \}, \qquad \mathbf{W} := \{ \alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha} \}.$$

Luk's theorem is a small-time theorem for large finite characteristic data. The data are not assumed small. The small parameter is the width of the spacetime slab $0 \le u \le \epsilon$, while the long direction $0 \le \underline{u} \le I$ is fixed by the initial data.

5.1 Schematic statement of the theorem

Theorem 5.1 (Luk's local existence theorem, schematic form)**.** *Let C_0 and $\underline{C}_0$ be two intersecting null hypersurfaces with $S_{0,0} = C_0 \cap \underline{C}_0$. Assume that regular characteristic initial data are prescribed on the initial null hypersurfaces and that the null constraint equations hold there. Let $\mathcal{I}$ denote the size of the initial data in the relevant regularity norms. Then there exists $\epsilon = \epsilon(\mathcal{I}) > 0$ such that the vacuum Einstein equations admit a unique solution, in the double-null gauge, in the slab*

$$0 \leq u \leq \epsilon, \qquad 0 \leq \underline{u} \leq I.$$

Moreover,

$$\mathcal{O}(u, \underline{u}) + \mathcal{R}(u, \underline{u}) \leq C(\mathcal{I})$$

throughout this slab, where $\mathcal{O}$ denotes the Ricci-coefficient norm and $\mathcal{R}$ denotes the curvature-flux norm.

Remark 5.2. Rendall's theorem gives a smooth solution in a small neighborhood of the corner $S_{0,0}$. This is used only to start the continuation argument. Luk's estimates then propagate the solution along the long initial direction and give a uniform slab $0 \leq u \leq \epsilon(\mathcal{I})$, $0 \leq \underline{u} \leq I$.

5.2 Schematic norms

We use simplified schematic norms, keeping only the structure needed to explain the proof. All angular derivatives and precise L^p-levels are suppressed. For

$$D_{u,\underline{u}} := \{0 \leq u' \leq u, \ 0 \leq \underline{u}' \leq \underline{u}\},$$

set

$$\mathcal{O}(u, \underline{u}) := \sup_{D_{u,\underline{u}}} |\Gamma|.$$

For curvature, set

$$\mathcal{R}(u, \underline{u})^2 := \sup_{0 \leq u' \leq u} \int_{C_{u'}(0,\underline{u})} \left(|\alpha|^2 + |\beta|^2 + \rho^2 + \sigma^2 + |\underline{\beta}|^2\right) + \sup_{0 \leq \underline{u}' \leq \underline{u}} \int_{\underline{C}_{\underline{u}'}(0,u)} \left(|\beta|^2 + \rho^2 + \sigma^2 + |\underline{\beta}|^2 + |\underline{\alpha}|^2\right).$$

The size of the initial data is denoted schematically by

$$\mathcal{I} := \mathcal{O}_{\text{init}} + \mathcal{R}_{\text{init}}.$$

The precise norms in Luk's theorem contain angular derivatives and several $L^p(S_{u,\underline{u}})$-levels. The simplified notation here is only meant to display the mechanism of the proof.

5.3 Main schematic equations

The proof uses the null structure equations

$$\nabla\!\!\!/_4 \Gamma = \mathbf{W} + \Gamma \cdot \Gamma, \qquad \nabla\!\!\!/_3 \Gamma = \mathbf{W} + \Gamma \cdot \Gamma,$$

and the Bianchi equations

$$\nabla\!\!\!/_3 \mathbf{W}_1 + d\!\!\!/^* \mathbf{W}_2 = \Gamma \cdot \mathbf{W}, \qquad \nabla\!\!\!/_4 \mathbf{W}_2 - d\!\!\!/ \mathbf{W}_1 = \Gamma \cdot \mathbf{W}.$$

The Bianchi pairs are

$$(\alpha, \beta), \qquad (\beta, (-\rho, \sigma)), \qquad ((\rho, \sigma), \underline{\beta}), \qquad (\underline{\beta}, \underline{\alpha}).$$

5.4 Transport estimates for Ricci coefficients

The favorable Ricci coefficient equations are the short-direction transport equations

$$\nabla\!\!\!/_3\Gamma = \mathbf{W} + \Gamma \cdot \Gamma.$$

Integrating along an e_3-generator gives

$$\Gamma(u, \underline{u}) = \Gamma(0, \underline{u}) + \int_0^u (\mathbf{W} + \Gamma \cdot \Gamma)(u', \underline{u})\, du'.$$

Thus, under the bootstrap assumption $\mathcal{O} \le M$,

$$\begin{aligned} |\Gamma(u, \underline{u})| &\le |\Gamma(0, \underline{u})| + \int_0^u |\mathbf{W}|(u', \underline{u})\, du' + \int_0^u |\Gamma|^2(u', \underline{u})\, du' \\ &\le |\Gamma(0, \underline{u})| + \epsilon^{1/2} \left(\int_0^u |\mathbf{W}|^2(u', \underline{u})\, du' \right)^{1/2} + \epsilon M^2. \end{aligned}$$

The factor $\epsilon^{1/2}$ comes from Cauchy's inequality in the short u-direction, and the term ϵM^2 is made harmless by choosing ϵ after M has been fixed.

The coefficients η and ω are exceptional. Their useful equations are in the long direction:

$$\nabla\!\!\!/_4\eta = \mathbf{W} + \Gamma \cdot \Gamma, \qquad \nabla\!\!\!/_4\omega = \mathbf{W} + \Gamma \cdot \Gamma.$$

Since the $\underline{u}$-length is I, integration in $\underline{u}$ gives no small factor. The key point is the absence of bad quadratic terms. Schematically, $\nabla\!\!\!/_4\eta$ contains no

$$\eta \cdot \eta, \qquad \eta\omega, \qquad \omega^2,$$

and, after η has been controlled, $\nabla\!\!\!/_4\omega$ contains no uncontrolled ω^2 term. Thus these long-direction equations are effectively linear once the good coefficients have been estimated.

Together with elliptic estimates on $S_{u,\underline{u}}$, the Ricci coefficient estimates give

$$\mathcal{O}(u, \underline{u}) \le C(\mathcal{I}) + C\epsilon^{1/2}\mathcal{R}(u, \underline{u}) + C\epsilon M^2.$$

5.5 Energy estimates for curvature

For one Bianchi pair,

$$\nabla\!\!\!/_3\mathbf{W}_1 + d\!\!\!/^*\mathbf{W}_2 = \Gamma \cdot \mathbf{W}, \qquad \nabla\!\!\!/_4\mathbf{W}_2 - d\!\!\!/\mathbf{W}_1 = \Gamma \cdot \mathbf{W}.$$

Multiplying by $2\mathbf{W}_1$ and $2\mathbf{W}_2$, respectively, gives

$$\begin{aligned} \nabla\!\!\!/_3|\mathbf{W}_1|^2 + \nabla\!\!\!/_4|\mathbf{W}_2|^2 = &-2\langle d\!\!\!/^*\mathbf{W}_2, \mathbf{W}_1\rangle + 2\langle d\!\!\!/\mathbf{W}_1, \mathbf{W}_2\rangle \\ &+ \Gamma \cdot \mathbf{W} \cdot \mathbf{W}. \end{aligned}$$

The angular terms cancel by formal adjointness:

$$2\langle d\!\!\!/\mathbf{W}_1, \mathbf{W}_2\rangle - 2\langle \mathbf{W}_1, d\!\!\!/^*\mathbf{W}_2\rangle = \mathrm{d}\!\!\!/\mathrm{iv}(\mathbf{W}_1 \cdot \mathbf{W}_2).$$

Thus

$$\nabla\!\!\!/_3|\mathbf{W}_1|^2 + \nabla\!\!\!/_4|\mathbf{W}_2|^2 = \mathrm{d}\!\!\!/\mathrm{iv}(\mathbf{W}_1 \cdot \mathbf{W}_2) + \Gamma \cdot \mathbf{W} \cdot \mathbf{W}.$$

Integrating over $D_{u,\underline{u}}$, the left-hand side gives the curvature fluxes on C_u and $\underline{C}_{\underline{u}}$, the initial sides give $\mathcal{R}_{\mathrm{init}}$, and

$$\int_{S_{u',\underline{u}'}} \mathrm{d}\!\!\!/\mathrm{iv}(\mathbf{W}_1 \cdot \mathbf{W}_2) = 0.$$

Hence

$$\mathcal{R}(u,\underline{u})^2 \lesssim \mathcal{R}_{\rm init}^2 + \int_{D_{u,\underline{u}}} |\Gamma|\,|\mathbf{W}|^2.$$

For the usual error terms, using $\mathcal{O} \leq M$,

$$\begin{aligned}\int_{D_{u,\underline{u}}} |\Gamma|\,|\mathbf{W}|^2 &\leq M \int_{D_{u,\underline{u}}} |\mathbf{W}|^2 \\ &\lesssim M\epsilon\,\mathcal{R}(u,\underline{u})^2.\end{aligned}$$

The factor ϵ comes from integrating the flux control in the short u-direction.

The α^2-terms are exceptional, because α is controlled on the outgoing null hypersurfaces. These terms are controlled by the favorable short-direction equation for the Ricci coefficient coupled to α^2, and the remaining contribution is handled by Gronwall. Thus the curvature estimate has the schematic form

$$\mathcal{R}(u,\underline{u})^2 \leq C\mathcal{R}_{\rm init}^2 + CM\epsilon\,\mathcal{R}(u,\underline{u})^2 + C(\mathcal{I}) \int_0^u \mathcal{R}(u',\underline{u})^2\,du'.$$

If $CM\epsilon \leq \frac{1}{2}$, the second term on the right is absorbed. Gronwall then gives

$$\mathcal{R}(u,\underline{u}) \leq A_2(\mathcal{I}),$$

where $A_2(\mathcal{I})$ is independent of M.

Substituting this curvature bound into the Ricci coefficient estimate gives

$$\mathcal{O}(u,\underline{u}) \leq C(\mathcal{I}) + C\epsilon M^2.$$

If $\epsilon M^2 \leq 1$, then

$$\mathcal{O}(u,\underline{u}) \leq A_1(\mathcal{I}),$$

where $A_1(\mathcal{I})$ is independent of M.

Set

$$A(\mathcal{I}) := A_1(\mathcal{I}) + A_2(\mathcal{I}).$$

Then, under the bootstrap assumption and the smallness conditions

$$CM\epsilon \leq \frac{1}{2}, \qquad \epsilon M^2 \leq 1,$$

we have the a priori estimate

$$\mathcal{O}(u,\underline{u}) + \mathcal{R}(u,\underline{u}) \leq A(\mathcal{I}),$$

and $A(\mathcal{I})$ is independent of M.

5.6 Continuation and closure

We now explain the bootstrap closure using only the schematic estimates needed for the argument. Assume that on $D_{\epsilon,\underline{u}_*}$ one has the bootstrap bound

$$\mathcal{O}(u,\underline{u}) + \mathcal{R}(u,\underline{u}) \leq M.$$

The estimates derived from the null structure and Bianchi equations have the schematic form

$$\mathcal{O}(u,\underline{u}) \lesssim_{\mathcal{I}} \mathcal{O}_{\rm init} + \epsilon^{1/2}M + \epsilon M^2,$$

and

$$\mathcal{R}(u,\underline{u})^2 \lesssim_{\mathcal{I}} \mathcal{R}_{\rm init}^2 + M^3\epsilon.$$

Here $\lesssim_{\mathcal{I}}$ means that the implicit constant depends only on the size of the initial data. The second estimate is written for $\mathcal{R}^2$, so the initial term is naturally $\mathcal{R}^2_{\text{init}}$.

Let $C_{\mathcal{I}}$ be a constant depending only on $\mathcal{I}$ such that

$$\mathcal{O}(u,\underline{u}) \le C_{\mathcal{I}} \left(\mathcal{O}_{\text{init}} + \epsilon^{1/2} M + \epsilon M^2\right),$$

and

$$\mathcal{R}(u,\underline{u})^2 \le C_{\mathcal{I}} \left(\mathcal{R}^2_{\text{init}} + M^3\epsilon\right).$$

Choose M first, depending only on the data, so large that

$$C_{\mathcal{I}}\mathcal{O}_{\text{init}} \le \frac{1}{16}M, \qquad C_{\mathcal{I}}\mathcal{R}^2_{\text{init}} \le \frac{1}{64}M^2.$$

Then choose $\epsilon = \epsilon(\mathcal{I}) > 0$ so small that

$$C_{\mathcal{I}}\epsilon^{1/2}M \le \frac{1}{16}M, \qquad C_{\mathcal{I}}\epsilon M^2 \le \frac{1}{16}M,$$

and

$$C_{\mathcal{I}} M^3 \epsilon \le \frac{1}{64}M^2.$$

Since M has already been fixed in terms of $\mathcal{I}$, these smallness conditions determine an ϵ depending only on $\mathcal{I}$.

With these choices,

$$\mathcal{O}(u,\underline{u}) \le \frac{3}{16}M,$$

and

$$\mathcal{R}(u,\underline{u})^2 \le \frac{1}{32}M^2, \qquad \text{hence} \qquad \mathcal{R}(u,\underline{u}) \le \frac{1}{4}M.$$

Therefore

$$\mathcal{O}(u,\underline{u}) + \mathcal{R}(u,\underline{u}) \le \frac{1}{2}M.$$

This strictly improves the bootstrap assumption.

Now define the continuation set

$$\mathcal{U}_M := \Big\{\underline{u}_* \in [0,I] : \text{the solution exists on } D_{\epsilon,\underline{u}_*},$$
$$\mathcal{O}(u,\underline{u}) + \mathcal{R}(u,\underline{u}) \le M \text{ for all } 0 \le u \le \epsilon,\ 0 \le \underline{u} \le \underline{u}_*\Big\}.$$

Rendall's theorem gives a smooth solution near $S_{0,0}$, so $\mathcal{U}_M \neq \emptyset$. Since the solution is smooth, the norms $\mathcal{O}$ and $\mathcal{R}$ are continuous in $(u,\underline{u})$, so $\mathcal{U}_M$ is closed in $[0,I]$. The openness step uses local characteristic existence restarted from the last sphere. Let $\underline{u}_* \in \mathcal{U}_M$. The a priori estimates give the strict improvement

$$\mathcal{O} + \mathcal{R} \le \frac{1}{2}M \qquad \text{on } D_{\epsilon,\underline{u}_*}.$$

Consider the sphere $S_{0,\underline{u}_*}$. To extend the solution beyond $\underline{u}_*$, use as characteristic data the prescribed initial data on $C_0(\underline{u}_*,\underline{u}_*+\delta)$, together with the induced data on the already constructed incoming hypersurface $\underline{C}_{\underline{u}_*}(0,\epsilon)$. These data satisfy the constraint equations because they either come from the original initial data or from the already constructed solution. The strict estimate above gives uniform control of their size. Therefore the local characteristic existence theorem gives a solution in a small neighborhood to the future of $S_{0,\underline{u}_*}$. Choosing $\delta > 0$ sufficiently small and using the continuity of the norms, the extended solution still satisfies

$$\mathcal{O} + \mathcal{R} \le M$$

on $D_{\epsilon,\underline{u}_*+\delta}$. Hence

$$\underline{u}_* + \delta \in \mathcal{U}_M,$$

so $\mathcal{U}_M$ is open. Hence

$$\mathcal{U}_M = [0, I].$$

Thus the solution exists on the whole slab

$$0 \le u \le \epsilon(\mathcal{I}), \qquad 0 \le \underline{u} \le I,$$

and satisfies

$$\mathcal{O}(u, \underline{u}) + \mathcal{R}(u, \underline{u}) \le M,$$

where $M = M(\mathcal{I})$.

In short,

$$\begin{aligned} \mathcal{I} < \infty &\Longrightarrow C_{\mathcal{I}} \\ &\Longrightarrow M = M(\mathcal{I}) \\ &\Longrightarrow \epsilon = \epsilon(M, \mathcal{I}) = \epsilon(\mathcal{I}) \\ \text{Rendall} &\Longrightarrow \mathcal{U}_M \neq \emptyset \\ \text{a priori estimates} &\Longrightarrow \mathcal{O} + \mathcal{R} \le \frac{1}{2} M \\ \text{continuity} &\Longrightarrow \mathcal{U}_M = [0, I]. \end{aligned}$$

6 Null decomposition formula sheet

$$L' := -2\mathbf{g}^{-1}(du), \qquad \underline{L}' := -2\mathbf{g}^{-1}(d\underline{u}),$$
$$\mathbf{g}(L', \underline{L}') =: -2\Omega^{-2}.$$

$$e_4 := \Omega L', \qquad e_3 := \Omega \underline{L}',$$
$$\mathbf{g}(e_3, e_4) = -2, \qquad \mathbf{g}(e_A, e_B) =: \not{g}_{AB}, \qquad \mathbf{g}(e_3, e_A) = \mathbf{g}(e_4, e_A) = 0.$$

$$L := \Omega e_4 = \Omega^2 L', \qquad \underline{L} := \Omega e_3 = \Omega^2 \underline{L}',$$
$$Lu = 0, \qquad L\underline{u} = 1, \qquad \underline{L}u = 1, \qquad \underline{L}\underline{u} = 0,$$
$$\mathbf{g}(L, \underline{L}) = -2\Omega^2.$$

$$D\Psi := (\mathcal{L}_L \Psi)^\top, \qquad \underline{D}\Psi := (\mathcal{L}_{\underline{L}} \Psi)^\top,$$
$$\not{\nabla}_4 \Psi := (\nabla_{e_4} \Psi)^\top, \qquad \not{\nabla}_3 \Psi := (\nabla_{e_3} \Psi)^\top.$$

6.1 Conventions

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$
$$\mathbf{Rm}(W, Z, X, Y) := \mathbf{g}(W, R(X,Y)Z).$$
$$\mathbf{Rm}_{\alpha\beta\gamma\delta} := \mathbf{Rm}(e_\alpha, e_\beta, e_\gamma, e_\delta).$$

$$\not{\mathrm{div}}\,\xi := \not{\nabla}^A \xi_A, \qquad \not{\mathrm{curl}}\,\xi := \not{\epsilon}^{AB} \not{\nabla}_A \xi_B, \qquad ({}^*\xi)_A := \not{\epsilon}_A{}^B \xi_B.$$

$$(\theta, \theta') := \not{g}^{AC} \not{g}^{BD} \theta_{AB} \theta'_{CD}, \qquad |\theta|^2 := (\theta, \theta).$$

$$(\theta\times\theta')_{AB} := \theta_A{}^C\theta'_{CB}, \qquad \theta\wedge\theta' := \not\epsilon^{AB}\theta_A{}^C\theta'_{BC}.$$

$$(\nabla\!\!\!/\widehat{\otimes}\xi)_{AB} := \nabla\!\!\!/_A\xi_B + \nabla\!\!\!/_B\xi_A - (\mathrm{d}\!\!/\mathrm{iv}\,\xi)\not g_{AB},$$
$$(\xi\widehat{\otimes}\xi)_{AB} := \xi_A\xi_B - \frac{1}{2}|\xi|^2\not g_{AB}.$$

In the structure and Bianchi equations below, we assume vacuum:

$$\mathbf{Ric} = 0, \qquad \mathbf{Rm} = \mathbf{W}.$$

6.2 Ricci coefficients

$$\chi_{AB} := \mathbf{g}(\nabla_A e_4, e_B), \qquad \underline{\chi}_{AB} := \mathbf{g}(\nabla_A e_3, e_B),$$
$$\hat\chi := \chi - \frac{1}{2}(\mathrm{tr}\chi)\not g, \qquad \hat{\underline{\chi}} := \underline{\chi} - \frac{1}{2}(\mathrm{tr}\underline{\chi})\not g.$$

$$\zeta_A := \frac{1}{2}\mathbf{g}(\nabla_A e_4, e_3),$$

$$\eta_A := \frac{1}{2}\mathbf{g}(\nabla_3 e_4, e_A), \qquad \underline{\eta}_A := \frac{1}{2}\mathbf{g}(\nabla_4 e_3, e_A),$$
$$\eta = \zeta + \not d\log\Omega, \qquad \underline{\eta} = -\zeta + \not d\log\Omega.$$

$$\omega := -\frac{1}{4}\mathbf{g}(\nabla_4 e_3, e_4), \qquad \underline{\omega} := -\frac{1}{4}\mathbf{g}(\nabla_3 e_4, e_3),$$
$$D\log\Omega = -2\Omega\omega, \qquad \underline{D}\log\Omega = -2\Omega\underline{\omega}.$$

$$\nabla_A e_4 = \chi_A{}^B e_B - \zeta_A e_4, \qquad \nabla_A e_3 = \underline{\chi}_A{}^B e_B + \zeta_A e_3,$$
$$\nabla_A e_B = \nabla\!\!\!/_A e_B + \frac{1}{2}\chi_{AB}e_3 + \frac{1}{2}\underline{\chi}_{AB}e_4,$$
$$\nabla_4 e_4 = -2\omega e_4, \qquad \nabla_3 e_3 = -2\underline{\omega}e_3,$$
$$\nabla_4 e_3 = 2\omega e_3 + 2\underline{\eta}^{\#}, \qquad \nabla_3 e_4 = 2\underline{\omega}e_4 + 2\eta^{\#}.$$

$$\nabla^{\perp}_A e_4 = -\zeta_A e_4, \qquad \nabla^{\perp}_A e_3 = \zeta_A e_3,$$
$$\frac{1}{2}\mathbf{g}(\nabla_A e_3, e_4) = -\zeta_A.$$

6.3 Weyl components

$$\alpha_{AB} := \mathbf{W}_{A4B4}, \qquad \underline{\alpha}_{AB} := \mathbf{W}_{A3B3},$$
$$\beta_A := \frac{1}{2}\mathbf{W}_{A434}, \qquad \underline{\beta}_A := \frac{1}{2}\mathbf{W}_{A334},$$
$$\rho := \frac{1}{4}\mathbf{W}_{3434}, \qquad \sigma := \frac{1}{4}\not\epsilon^{AB}\mathbf{W}_{AB34}.$$

$$\mathbf{W}_{A434} = 2\beta_A, \qquad \mathbf{W}_{A334} = 2\underline{\beta}_A, \qquad \mathbf{W}_{3434} = 4\rho, \qquad \mathbf{W}_{AB34} = 2\sigma\not\epsilon_{AB}.$$
$$\mathbf{W}_{A4B3} = -\rho\not g_{AB} - \sigma\not\epsilon_{AB}.$$

$$\Gamma := \{\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \omega, \underline{\omega}\}, \qquad \mathbf{W} := \{\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}.$$

6.4 Double-null identities

$$\eta = \zeta + \not{d}\log\Omega, \qquad \underline{\eta} = -\zeta + \not{d}\log\Omega,$$
$$\zeta = \frac{1}{2}(\eta - \underline{\eta}), \qquad \not{d}\log\Omega = \frac{1}{2}(\eta + \underline{\eta}).$$

$$D\log\Omega = -2\Omega\omega, \qquad \underline{D}\log\Omega = -2\Omega\underline{\omega},$$
$$D\Omega = -2\Omega^2\omega, \qquad \underline{D}\Omega = -2\Omega^2\underline{\omega}.$$

$$[e_4, e_3] = 2(\underline{\eta} - \eta)^\# + 2\omega e_3 - 2\underline{\omega} e_4.$$

$$[L, \underline{L}] = -4\Omega^2\zeta^\#.$$

$$\underline{L} = \partial_u, \qquad L = \partial_{\underline{u}} + b^A\partial_A,$$
$$[L, \underline{L}] = -(\partial_u b^A)\partial_A, \qquad \partial_u b^A = 4\Omega^2\zeta^A.$$

$$D\not{d}f = \not{d}Df, \qquad \underline{D}\not{d}f = \not{d}\,\underline{D}f.$$

$$D(\underline{D}f) - \underline{D}(Df) = -4\Omega^2\zeta^A\not{\nabla}_A f.$$

6.5 Null structure equations

$$D\not{g} = 2\Omega\chi, \qquad \underline{D}\not{g} = 2\Omega\underline{\chi},$$
$$D(d\mu_{\not{g}}) = \Omega\mathrm{tr}\chi\, d\mu_{\not{g}}, \qquad \underline{D}(d\mu_{\not{g}}) = \Omega\mathrm{tr}\underline{\chi}\, d\mu_{\not{g}}.$$

$$D\chi = -2\Omega\omega\,\chi + \Omega\chi\times\chi - \Omega\alpha,$$
$$\underline{D}\underline{\chi} = -2\Omega\underline{\omega}\,\underline{\chi} + \Omega\underline{\chi}\times\underline{\chi} - \Omega\underline{\alpha}.$$

$$D\mathrm{tr}\chi = -\frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \Omega|\hat{\chi}|^2 - 2\Omega\omega\,\mathrm{tr}\chi,$$
$$\underline{D}\mathrm{tr}\underline{\chi} = -\frac{\Omega}{2}(\mathrm{tr}\underline{\chi})^2 - \Omega|\underline{\hat{\chi}}|^2 - 2\Omega\underline{\omega}\,\mathrm{tr}\underline{\chi}.$$

$$D\hat{\chi} = -\Omega(\mathrm{tr}\chi)\hat{\chi} - 2\Omega\omega\,\hat{\chi} - \Omega\alpha,$$
$$\underline{D}\underline{\hat{\chi}} = -\Omega(\mathrm{tr}\underline{\chi})\underline{\hat{\chi}} - 2\Omega\underline{\omega}\,\underline{\hat{\chi}} - \Omega\underline{\alpha}.$$

$$D(\Omega\underline{\chi})_{AB} = \Omega^2\Big(\not{\nabla}_A\eta_B + \not{\nabla}_B\eta_A + 2\eta_A\eta_B + \frac{1}{2}(\chi_A{}^C\underline{\chi}_{CB} + \underline{\chi}_A{}^C\chi_{CB}) + \rho\,\not{g}_{AB}\Big),$$
$$\underline{D}(\Omega\chi)_{AB} = \Omega^2\Big(\not{\nabla}_A\underline{\eta}_B + \not{\nabla}_B\underline{\eta}_A + 2\underline{\eta}_A\underline{\eta}_B + \frac{1}{2}(\chi_A{}^C\underline{\chi}_{CB} + \underline{\chi}_A{}^C\chi_{CB}) + \rho\,\not{g}_{AB}\Big).$$

$$D(\Omega\mathrm{tr}\underline{\chi}) = \Omega^2\left(2\not{\mathrm{d}}\mathrm{iv}\eta + 2|\eta|^2 + (\chi,\underline{\chi}) + 2\rho\right),$$
$$\underline{D}(\Omega\mathrm{tr}\chi) = \Omega^2\left(2\not{\mathrm{d}}\mathrm{iv}\underline{\eta} + 2|\underline{\eta}|^2 + (\chi,\underline{\chi}) + 2\rho\right).$$

$$(D\zeta)_A = 2(\not{d}(\Omega\omega))_A + \Omega\chi_A{}^B\eta_B - \Omega\beta_A,$$
$$(\underline{D}\zeta)_A = -2(\not{d}(\Omega\underline{\omega}))_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B - \Omega\underline{\beta}_A.$$

$$(D\eta)_A - \Omega\chi_A{}^B\eta_B = -\Omega\beta_A,$$
$$(\underline{D}\underline{\eta})_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B = \Omega\underline{\beta}_A.$$

$$(D\underline{\eta})_A = -4(\not{d}(\Omega\omega))_A - \Omega\chi_A{}^B\eta_B + \Omega\beta_A,$$
$$(\underline{D}\eta)_A = -4(\not{d}(\Omega\underline{\omega}))_A - \Omega\underline{\chi}_A{}^B\underline{\eta}_B - \Omega\underline{\beta}_A.$$

$$D\omega = 2\Omega\omega^2 - \frac{\Omega}{2}\left(2(\eta,\underline{\eta}) - |\eta|^2 - \rho\right),$$
$$\underline{D}\underline{\omega} = 2\Omega\underline{\omega}^2 - \frac{\Omega}{2}\left(2(\eta,\underline{\eta}) - |\underline{\eta}|^2 - \rho\right).$$

$$D\underline{\omega} = \Omega\left(2\omega\underline{\omega} - (\eta,\underline{\eta}) + \frac{1}{2}|\eta|^2 - \frac{1}{2}\rho\right),$$
$$\underline{D}\omega = \Omega\left(2\omega\underline{\omega} - (\eta,\underline{\eta}) + \frac{1}{2}|\underline{\eta}|^2 - \frac{1}{2}\rho\right).$$

$$K = -\rho - \frac{1}{4}(\mathrm{tr}\chi)(\mathrm{tr}\underline{\chi}) + \frac{1}{2}(\hat{\chi},\hat{\underline{\chi}}).$$

$$(\not{\mathrm{div}}\chi)_A - (\not{d}\mathrm{tr}\chi)_A + \chi_A{}^B\zeta_B - (\mathrm{tr}\chi)\zeta_A = -\beta_A,$$
$$(\not{\mathrm{div}}\underline{\chi})_A - (\not{d}\mathrm{tr}\underline{\chi})_A - \underline{\chi}_A{}^B\zeta_B + (\mathrm{tr}\underline{\chi})\zeta_A = \underline{\beta}_A.$$

$$\not{\mathrm{curl}}\zeta = \sigma - \frac{1}{2}\chi\wedge\underline{\chi},$$
$$\not{\mathrm{curl}}\eta = \sigma - \frac{1}{2}\chi\wedge\underline{\chi}, \qquad \not{\mathrm{curl}}\underline{\eta} = -\sigma + \frac{1}{2}\chi\wedge\underline{\chi}.$$

6.6 Bianchi equations

$$\nabla^\alpha \mathbf{W}_{\alpha\beta\gamma\delta} = 0.$$

$$\not{d}_1\xi := (\not{\mathrm{div}}\xi, \not{\mathrm{curl}}\xi), \qquad \not{d}_1^*(f,g) := -\not{d}f + {}^*\not{d}g.$$
$$(\not{d}_2\theta)_A := \not{\nabla}^B\theta_{AB}, \qquad (\not{d}_2^*\xi)_{AB} := -\frac{1}{2}\left(\not{\nabla}_A\xi_B + \not{\nabla}_B\xi_A - (\not{\mathrm{div}}\xi)\not{g}_{AB}\right).$$

$$\not{\nabla}_3\alpha + \not{d}_2^*\beta = \Gamma\cdot\mathbf{W}, \qquad \not{\nabla}_4\beta - \not{d}_2\alpha = \Gamma\cdot\mathbf{W},$$
$$\not{\nabla}_3\beta - \not{d}_1^*(-\rho,\sigma) = \Gamma\cdot\mathbf{W}, \qquad \not{\nabla}_4(-\rho,\sigma) + \not{d}_1\beta = \Gamma\cdot\mathbf{W},$$
$$\not{\nabla}_4\underline{\beta} + \not{d}_1^*(\rho,\sigma) = \Gamma\cdot\mathbf{W}, \qquad \not{\nabla}_3(\rho,\sigma) - \not{d}_1\underline{\beta} = \Gamma\cdot\mathbf{W},$$
$$\not{\nabla}_4\underline{\alpha} + \not{d}_2^*\underline{\beta} = \Gamma\cdot\mathbf{W}, \qquad \not{\nabla}_3\underline{\beta} - \not{d}_2\underline{\alpha} = \Gamma\cdot\mathbf{W}.$$

References

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